Arithmetic geometry of algebraic curves and their moduli space

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Abstract

We review the following subjects:

- Basic theory on algebraic curves and their moduli space,

- Schottky uniformization theory of Riemann surfaces, and its extension called arithmetic uniformization theory,

- Application to these theories to the arithmetic of the moduli space of algebraic curves, especially to automorphic forms on this space.

Contents

§1. Introduction

1.1. Brief history

- Around 1800–1830, Gauss(1777–1855), Abel(1802–1829) and Jacobi(1804–1851) found that the inverse function of the elliptic integral:

$$y \;=\; \int dx \Big/ \sqrt{f(x)} \;\; (f(x) : \text{a polynomial of degree 4 without multiple root})$$

is an elliptic function, i.e., a double periodic function of the complex variable y, and they expressed the function as an infinite product and the ratios of theta functions.
$\Rightarrow$ complex function theory.

- Riemann(1826–1866) constructed Riemann surfaces from algebraic function fields

$$\mathbf{C}(x, y) \;\; (x : \text{a variable}, \; y : \text{finite over } \mathbf{C}(x)),$$

and solved Jacobi's inverse problem using Abel-Jacobi's theorem and Riemann's theta functions.
$\Rightarrow$ complex geometry and algebraic geometry (1857).

- Teichmüller(1913–1943) constructed analytic theory on the moduli of Riemann surfaces.

- Mumford constructed the moduli of algebraic curves as an algebraic variety (1956), and studied this geometry. Further, he and Deligne gave its compactification as the moduli of stable curves (1969).

- String theory provided a strong relationship between physics and the theory of moduli of curves.

- Around 1960–1970, Shimura constructed arithmetic theory on Shimura models with applications to the rationality on Siegel modular forms, and further Chai and Faltings extended his result to any base ring (1990).

1.2. Plan of this lecture

We will review the following subjects with some proof:

- Very classical results on algebraic curves over $\mathbf{C}$ and the associated Riemann surfaces: for example, $\wp$-functions and elliptic curves, differential 1-forms and period integrals, Riemann-Roch's theorem, Abel-Jacobi's theorem and Jacobian varieties, degeneration, Schottky uniformization and the description of forms and periods.

- Rather classical results on moduli and families of algebraic curves: for example, moduli of elliptic curves and higher genus curves, stable curves and their moduli (Deligne-Mumford's compactification), the irreducibility of the moduli, Eisenstein series and Tate curve, Mumford curves;

  and recent results on arithmetic version of Schottky uniformization.

- Recent results on arithmetic geometry of the moduli space of algebraic curves: for example, Fourier expansion of (elliptic and Siegel) modular forms and their rationality, Teichmüller modular forms and the Schottky problem, Mumford's isomorphism.

Especially, we explain that the classical, but not so familiar Schottky uniformization theory which gives an explicit description of differential forms, periods and degeneration of Riemann surfaces. Furthermore, we give its extended version in the category of arithmetic geometry (unifying complex geometry and formal geometry over $\mathbf{Z}$), and the application to automorphic forms, called Teichmüller modular forms, on the moduli space of algebraic curves.

<h1 style="text-align:center">§2. Algebraic curves and Riemann surfaces</h1>
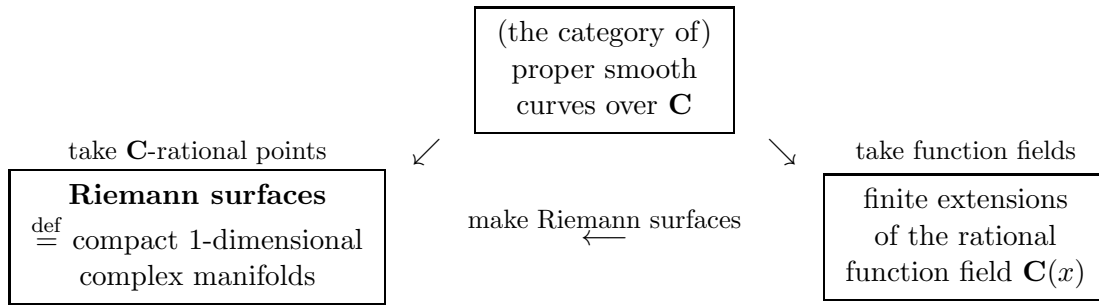
2.1. Riemann's correspondence

Algebraic curves. **Algebraic varieties** are topological spaces obtained by gluing zero sets of polynomials of multiple variables, and closed subsets of algebraic varieties are defined as zero sets of polynomials (Zariski topology). These examples are

$$\text{the } \textbf{projective } \boldsymbol{n}\textbf{-space } \ \mathbb{P}^n_k \ \overset{\text{def}}{=} \ (k^{n+1} - \{(0,...,0)\})/k^\times$$
$$= \ \{(x_0 : \cdots : x_{n+1}) = (cx_0 : \cdots : cx_{n+1}) \mid c \neq 0\},$$

and its subsets (called **projective varieties** which are **proper** over k ($\doteq$ compact)) as the zero sets of homogeneous polynomials over an algebraically closed field k.
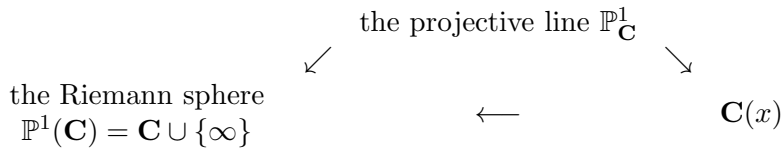
$$\text{(algebraic) } \textbf{curves} \overset{\text{def}}{=} \text{1-dimensional algebraic varieties}$$

Riemann's correspondence. There exists an equivalence (trinity) of the categories:
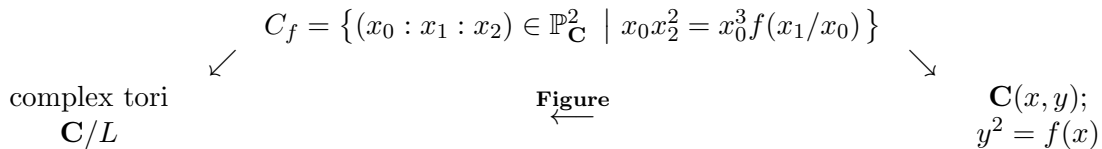
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(the category of)
proper smooth
curves over **C**

</div>

take **C**-rational points $\swarrow$ $\searrow$ take function fields

<div style="text-align:center">

Riemann surfaces
$\overset{\text{def}}{=}$ compact 1-dimensional
complex manifolds

make Riemann surfaces $\longleftarrow$

finite extensions
of the rational
function field **C**(x)

</div>

Genus. The **genus** of a Riemann surface and the corresponding curve is defined as the number of its holes (**Figure**).

Genus 0 case.

<div style="text-align:center">

the projective line $\mathbb{P}^1_{\mathbf{C}}$

$\swarrow$ $\searrow$

the Riemann sphere
$\mathbb{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$ $\longleftarrow$ $\mathbf{C}(x)$

</div>

Genus 1 case. For cubic polynomials $f(x) \in \mathbf{C}[x]$ without multiple root,

$$C_f = \left\{ (x_0 : x_1 : x_2) \in \mathbb{P}^2_{\mathbf{C}} \ \middle| \ x_0 x_2^2 = x_0^3 f(x_1/x_0) \right\}$$

<div style="text-align:center">

$\swarrow$ $\searrow$

complex tori $\qquad$ **Figure** $\qquad$ $\mathbf{C}(x, y);$
$\mathbf{C}/L$ $\qquad\quad$ $\longleftarrow$ $\qquad\quad$ $y^2 = f(x)$

</div>

Here

L is a **lattice** in $\mathbf{C}$, i.e., a sub **Z**-module of rank 2 such that $L \otimes_{\mathbf{Z}} \mathbf{R} = \mathbf{C}$,

$E_{2k}(L) \overset{\text{def}}{=} \displaystyle\sum_{u \in L - \{0\}} \frac{1}{u^{2k}}$: absolutely convergent series for $k > 1$,

$$f(x) \overset{\text{def}}{=} 4x^3 - 60E_4(L)x - 140E_6(L),$$

$$\wp(z) = \wp_L(z) \overset{\text{def}}{=} \frac{1}{z^2} + \sum_{u \in L - \{0\}} \left(\frac{1}{(z-u)^2} - \frac{1}{u^2} \right) : \text{ Weierstrass' } \wp\text{-function}$$

$$\Rightarrow \begin{cases} \wp(z) \text{ is absolutely and uniformly convergent on any compact subset of } \mathbf{C} - L, \\ z \mapsto (1 : \wp(z) : \wp'(z)) \text{ gives a biholomorphic map } \mathbf{C}/L \overset{\sim}{\to} C_f(\mathbf{C}), \end{cases}$$

and

$$\mathbf{C}(x, y) \text{ is a quadratic extension of } \mathbf{C}(x) \text{ defined by } y^2 = f(x)$$

$$\leftrightarrow \quad C_f \text{ is a double cover of } \mathbb{P}^1_{\mathbf{C}} \text{ ramified at the 3 roots of } f(x) \text{ and } \infty.$$

An **elliptic curve** is a proper smooth curve C of genus 1 and with one marked point x_0. Then C has unique commutative group structure defined algebraically with origin x_0. For example, the above C_f with $(0 : 0 : 1)$ is an elliptic curve, and the map $\mathbf{C}/L \overset{\sim}{\to} C_f(\mathbf{C})$ is also a group isomorphism which follows from the addition law of $\wp(z)$:

$$\wp(z + w) = -\wp(z) - \wp(w) + \frac{1}{4} \left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} \right)^2.$$

Exercise 1. Show the following Laurent expansion of $\wp(z)$:

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n + 1)E_{2n+2}(L)z^{2n} \quad \text{around } z = 0.$$

Further, using this fact, the periodicity of $\wp(z)$:

$$\wp(z + u) = \wp(z) \quad (u \in L),$$

and the maximum principle on holomorphic functions, prove that

$$\wp'(z)^2 = 4\wp(z)^3 - 60E_4(L)\wp(z) - 140E_6(L)$$

$$\left(\text{i.e., } \wp(z) = x \ \Rightarrow \ z = \int \frac{dx}{\sqrt{4x^3 - 60E_4 x - 140E_6}} \right)$$

and that $E_8(L) = \frac{3}{7}E_4(L)^2$.

Genus > 1 case. For a Riemann surface R of genus > 1, by Riemann's mapping theorem, its universal cover is biholomorphic to

$$H_1 \overset{\text{def}}{=} \{ \tau \in \mathbf{C} \mid \text{Im}(\tau) > 0 \} : \text{the } \textbf{Poincaré upper half plane}.$$

Then we have

$$R \cong H_1/\pi_1(R) : \text{ called a } \textbf{Fuchsian model},$$

where the fundamental group $\pi_1(R)$ of R is a cocompact discrete subgroup of

$$PSL_2(\mathbf{R}) \overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{R}) \ \middle| \ ad - bc = 1 \right\} \bigg/ \{\pm E_2\}$$

which acts on H_1 by the Möbius transformation:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

(in fact, $PSL_2(\mathbf{R})$ is the group $\mathrm{Aut}(H_1)$ of complex analytic automorphisms of H_1).

Remark. Let Γ be a congruence subgroup of $SL_2(\mathbf{Z})$, for example

$$SL_2(\mathbf{Z}) \overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}) \ \middle|\ ad - bc = 1 \right\}$$

$$\supset \Gamma_0(N) \overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \ \middle|\ c \equiv 0 \bmod(N) \right\}$$

$$\supset \Gamma(N) \overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \ \middle|\ a - 1 \equiv b \equiv c \equiv d - 1 \equiv 0 \bmod(N) \right\}$$

$$: \quad \text{the principal congruence subgroup of } SL_2(\mathbf{Z}) \text{ of level } N.$$

Then H_1/Γ is a *noncompact* 1-dimensional complex manifold, and becomes compact by adding the set $\mathbb{P}^1(\mathbf{Q})/\Gamma$ of **cusps** of Γ. H_1/Γ and $\left(H_1 \cup \mathbb{P}^1(\mathbf{Q}) \right)/\Gamma$ are called **modular curves**.

2.2. Riemann-Roch's theorem

Let R be a Riemann surface of genus g, and let D be a divisor on R which is, by definition, a finite sum of points on R with coefficients in $\mathbf{Z}$. When D is represented as $\sum_{P \in R} a_P \cdot P$, the associated invertible sheaf $\mathcal{O}_R(D)$, namely line bundle, on R is defined as

$$\mathcal{O}_R(D)(U) \overset{\text{def}}{=} \{f \colon \text{meromorphic functions on } U \mid \mathrm{ord}_P(f) + a_P \geqq 0 \ (P \in U)\}$$

for open subsets U of R. Then Riemann-Roch's theorem states

$$\dim_{\mathbf{C}} H^0(R, \mathcal{O}_R(D)) - \dim_{\mathbf{C}} H^1(R, \mathcal{O}_R(D)) = \deg(D) + 1 - g,$$

where $\deg(D) = \sum_{P \in R} a_P$ is the degree of D. Denote by Ω_R the invertible sheaf of holomorphic 1-forms on R. Then by Serre's duality, the residue map gives a nondegenerate pairing

$$H^0(R, \Omega_R(-D)) \times H^1(R, \mathcal{O}_R(D)) \to \mathbf{C},$$

and hence

$$\dim_{\mathbf{C}} H^0(R, \mathcal{O}_R(D)) - \dim_{\mathbf{C}} H^0(R, \Omega_R(-D)) = \deg(D) + 1 - g.$$

2.3. Differential forms, periods and Jacobians

Let R be a Riemann surface of genus $g \geq 1$. Then its fundamental group $\pi_1(R; x_0)$ with base point $x_0 \in R$ is represented by

$$\left\langle \underbrace{\alpha_1, \beta_1, ..., \alpha_g, \beta_g}_{\text{generators}} \ \middle|\ \underbrace{\prod_{i=1}^{g} (\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}) = 1}_{\text{relation}} \right\rangle,$$

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where the generators α_i, β_i are **canonical**, i.e., closed oriented paths in R with base point x_0 such that α_i, β_i intersect as the x, y-axes and that $(\alpha_i \cup \beta_i) \cap (\alpha_j \cup \beta_j) = \{x_0\}$ if $i \neq j$ (**Figure**). Then

Theorem 2.1. (Abel, Jacobi, Riemann, see [Mur])

(1) *The space $H^0(R, \Omega_R)$ of* **holomorphic 1-forms** *on R is g-dimensional, and is generated by unique holomorphic 1-forms $\omega_1, ..., \omega_g$ satisfying that $\int_{\alpha_i} \omega_j = \delta_{ij}$. Furthermore,* $\deg(\Omega_R) = 2g - 2$.

(2) *(Riemann's period relation) The* **period matrix**

$$Z \overset{\text{def}}{=} \left(\int_{\beta_i} \omega_j \right)_{1 \leq i, j \leq g}$$

of $(R; \{\alpha_i, \beta_i\}_{1 \leq i \leq g})$ is symmetric, and its imaginary part $\mathrm{Im}(Z)$ is positive definite.

(3) *(Abel-Jacobi's theorem) Let*

$$\mathrm{Cl}^0(R) \;=\; \{\textit{divisors with degree } 0 \textit{ on } R\} \Big/ \left\{ \sum_{P \in R} \mathrm{ord}_P(f) \cdot P \right\}$$

the **divisor class group** *with degree 0 of R, and let $\mathbf{C}^g / L$ be the g-dimensional complex torus obtained from the lattice $L \overset{\text{def}}{=} \mathbf{Z}^g + \mathbf{Z}^g \cdot Z$ in $\mathbf{C}^g$. Then the map*

$$\sum_j (P_j - Q_j) \mapsto \left(\sum_j \int_{Q_j}^{P_j} \omega_i \right)_{1 \leq i \leq g}$$

gives rise to a group isomorphism:

$$\mu : \mathrm{Cl}^0(R) \overset{\sim}{\longrightarrow} \mathbf{C}^g / L.$$

Remark. It is clear that $(z_1, ..., z_g) \mapsto \left(\exp(2\pi\sqrt{-1}z_1), ..., \exp(2\pi\sqrt{-1}z_g) \right)$ gives the isomorphism

$$\mathbf{C}^g / L \overset{\sim}{\to} (\mathbf{C}^\times)^g \Big/ \left\langle \left(\exp\left(2\pi\sqrt{-1} \int_{\beta_i} \omega_j \right) \right)_{1 \leq i \leq g} \;\bigg|\; 1 \leq j \leq g \right\rangle,$$

and then

$$\exp\left(2\pi\sqrt{-1} \int_{\beta_i} \omega_j \right) \quad (1 \leq i, j \leq g)$$

are called the **multiplicative periods**. Let $\mathrm{Pic}^0(R)$ denote the **Picard group** with degree 0 of R which is defined as the group of linear equivalence classes of line bundles with degree 0 over R. Then it is known that

$$\begin{aligned} \mathrm{Cl}^0(R) \;&\cong\; \mathrm{Pic}^0(R) \\ D \;&\leftrightarrow\; \mathcal{O}_R(D) \end{aligned}$$

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becomes an **abelian variety**, i.e., a proper (commutative) algebraic group over $\mathbf{C}$, and the isomorphism is also a biholomorphic map. This abelian variety is called the **Jacobian variety** of R (or of the associated curve), and denoted by $\mathrm{Jac}(R)$.

Proof. (1) By Riemann-Roch's theorem,

$$
\begin{aligned}
\dim_{\mathbf{C}} H^0(R, \mathcal{O}_R) - \dim_{\mathbf{C}} H^0(R, \Omega_R) &= 1 - g, \\
\dim_{\mathbf{C}} H^0(R, \Omega_R) - \dim_{\mathbf{C}} H^0(R, \mathcal{O}_R) &= \deg(\Omega_R) + 1 - g.
\end{aligned}
$$

By the maximum principle on holomorphic functions on R, $H^0(R, \mathcal{O}_R)$ consists of constant functions on R, and hence

$$
\dim_{\mathbf{C}} H^0(R, \Omega_R) = g, \ \deg(\Omega_R) = 2g - 2.
$$

To prove the remains of (1), and (2), (3), first we show a generalized form of Riemann's period relation. Let $\mathcal{P}$ the $4g$ oriented sided polygon obtained from R by cutting the paths α_i, β_i ($1 \le i \le g$) (**Figure**). Fix $P_0 \in \mathcal{P}$, and for a holomorphic 1-form ϕ on R, define $f(P) = \int_{P_0}^P \phi$. Then for a meromorphic 1-form ψ on R whose poles belong to the interior $\mathcal{P}^\circ$ of $\mathcal{P}$ (this condition is satisfied by moving slightly α_i, β_i if necessary), using the function $f^\pm$ on the boundary $\partial\mathcal{P}$ of $\mathcal{P}$ defined by

$$
f^+(P) \ \overset{\mathrm{def}}{=} \ \int_{P_0}^P \phi \ \ (P \in \alpha_i \cup \beta_i),
$$

$$
f^-(P) \ \overset{\mathrm{def}}{=} \ \int_{P_0}^P \phi \ \ (P \in -\alpha_i \cup -\beta_i),
$$

we have

$$
\begin{aligned}
2\pi\sqrt{-1} \sum_{P \in \mathcal{P}} \mathrm{Res}_P(f\psi) \ &= \ \int_{\partial\mathcal{P}} f\psi \ \ \text{(by the residue theorem)} \\
&= \ \sum_{i=1}^g \left(\int_{\alpha_i} f^+\psi + \int_{-\alpha_i} f^-\psi + \int_{\beta_i} f^+\psi + \int_{-\beta_i} f^-\psi \right) \\
&= \ \sum_{i=1}^g \left(\int_{\alpha_i} (f^+ - f^-)\psi + \int_{\beta_i} (f^+ - f^-)\psi \right) \\
&= \ \sum_{i=1}^g \left(\left(-\int_{\beta_i} \phi \right) \left(\int_{\alpha_i} \psi \right) + \left(\int_{\alpha_i} \phi \right) \left(\int_{\beta_i} \psi \right) \right).
\end{aligned}
$$

Therefore,

$$
2\pi\sqrt{-1} \sum_{P \in \mathcal{P}} \mathrm{Res}_P(f\psi) \ = \ \sum_{i=1}^g \left(\left(\int_{\alpha_i} \phi \right) \left(\int_{\beta_i} \psi \right) - \left(\int_{\beta_i} \phi \right) \left(\int_{\alpha_i} \psi \right) \right)
$$

which we call the **generalized Riemann's period relation**.

In particular, for two holomorphic 1-forms φ, φ', put $f(P) = \int_{P_0}^{P} \varphi$ $(P \in \mathcal{P}^{\circ})$, and put

$$A_i = \int_{\alpha_i} \varphi, \quad A_i' = \int_{\alpha_i} \varphi', \quad B_i = \int_{\beta_i} \varphi, \quad B_i' = \int_{\beta_i} \varphi'.$$

Then by the above,

$$\sum_{i=1}^{g} \left(A_i B_i' - B_i A_i' \right) = 0.$$

Further,

$$\begin{aligned}
\operatorname{Im}\left(\sum_{i=1}^{g} \overline{A_i} B_i \right) &= \frac{1}{2\sqrt{-1}} \sum_{i=1}^{g} \left(\overline{A_i} B_i - \overline{B_i} A_i \right) \\
&= \frac{1}{2\sqrt{-1}} \int_{\partial \mathcal{P}} \overline{f} \varphi = \frac{1}{2\sqrt{-1}} \int_{\mathcal{P}} d\left(\overline{f} \varphi \right) \\
&= \int_{\mathcal{P}} du\, dv = \int_{\mathcal{P}} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) dx\, dy \\
&\quad \left(f = u + \sqrt{-1}v, \ z = x + \sqrt{-1}y : \text{local coordinates} \right)
\end{aligned}$$

by Cauchy-Riemann's relation $\partial u / \partial x = \partial v / \partial y$, $\partial u / \partial y = -\partial v / \partial x$, and hence this value is positive if φ is not identically 0. Therefore, any holomorphic 1-form φ with $\int_{\alpha_i} \varphi = 0$ $(1 \le i \le g)$ becomes identically 0, and hence for any base ω_i' $(1 \le i \le g)$ of $H^0(R, \Omega_R)$, $\left(\int_{\alpha_i} \omega_j' \right)_{i,j}$ is a regular matrix. This implies (1).

(2) If $\varphi = \omega_i, \varphi' = \omega_j$, then by the above, $\int_{\beta_i} \omega_j - \int_{\beta_j} \omega_i = 0$, hence Z is symmetric. Further, if $\varphi = \sum_{i=1}^{g} c_i \omega_i \in H^0(R, \Omega_R)$ is not 0, then $\boldsymbol{c} = (c_1, ..., c_g) \neq \boldsymbol{0}$, and hence $\operatorname{Im}(\overline{\boldsymbol{c}} Z^t \boldsymbol{c}) > 0$. This implies that $\operatorname{Im}(Z)$ is positive definite. This proves (2).

(3) If f is a meromorphic function on R, then by the generalized period relation,

$$\begin{aligned}
&\sum_{P \in R} \left(\operatorname{ord}_P(f) \cdot \int_{P_0}^{P} \omega_j \right) \\
&= \sum_{P \in \mathcal{P}} \operatorname{Res}_P \left(\int_{P_0}^{P} \omega_j \cdot \frac{df}{f} \right) \\
&= \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{g} \left(\left(\int_{\alpha_i} \omega_j \right) \left(\int_{\beta_i} \frac{df}{f} \right) - \left(\int_{\beta_i} \omega_j \right) \left(\int_{\alpha_i} \frac{df}{f} \right) \right) \\
&\in L
\end{aligned}$$

because $\int_{\alpha_i} df/f, \int_{\beta_i} df/f \in 2\pi\sqrt{-1}\mathbf{Z}$. Hence the map μ in (3) is well-defined.

Next, we show the injectivity of μ. By Riemann-Roch's theorem, for $P_1, P_2 \in R$,

$$\dim_{\mathbf{C}} H^0(R, \Omega_R(P_1 + P_2)) = g + 1 = \dim_{\mathbf{C}} H^0(R, \Omega_R) + 1,$$

and hence there is a meromorphic 1-form on R which is holomorphic except for simple poles at P_1, P_2 with residues $1, -1$ respectively. Let D be a divisor with degree 0 on R such that

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$\mu(D) \in L$. Then D is represented as a finite sum $\sum_i a_i \left(P_1^{(i)} - P_2^{(i)} \right)$ for $a_i \in \mathbf{Z}$, $P_1^{(i)}, P_2^{(i)} \in R$, and hence there is a meromorphic 1-form ψ on R such that $\sum_{P \in R} \mathrm{Res}_P(\psi) \cdot P = D$. Hence by the period relation,

$$
\begin{aligned}
\mu(D) &= \left(\sum_{P \in \mathcal{P}} \mathrm{Res}_P \left(\int_{P_0}^{P} \omega_j \cdot \psi \right) \right)_{1 \leq j \leq g} \\
&= \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{g} \left(\left(\int_{\alpha_i} \omega_j \right) \left(\int_{\beta_i} \psi \right) - \left(\int_{\beta_i} \omega_j \right) \left(\int_{\alpha_i} \psi \right) \right)_{1 \leq j \leq g} \\
&\in L = \left\{ \sum_{i=1}^{g} \left(m_i \int_{\alpha_i} \omega_j - l_i \int_{\beta_i} \omega_j \right)_{1 \leq j \leq g} \ \middle| \ m_i, l_i \in \mathbf{Z} \right\}.
\end{aligned}
$$

Therefore, there are integers m_i, l_i $(1 \leq i \leq g)$ such that

$$
\sum_{i=1}^{g} \left(\left(\int_{\beta_i} \psi - (2\pi\sqrt{-1}) m_i \right) \left(\int_{\alpha_i} \omega_j \right) - \left(\int_{\alpha_i} \psi - (2\pi\sqrt{-1}) l_i \right) \left(\int_{\beta_i} \omega_j \right) \right) = 0
$$

for any $1 \leq j \leq g$. By (1), the orthogonal subspace of $\mathbf{C}^{2g}$ to

$$
\left(\int_{\alpha_1} \omega_j, \cdots, \int_{\alpha_g} \omega_j, \int_{\beta_1} \omega_j, \cdots, \int_{\beta_g} \omega_j \right) \quad (1 \leq j \leq g)
$$

has dimension g, and by the period relation, this is generated by

$$
\left(\int_{\beta_1} \omega_j, \cdots, \int_{\beta_g} \omega_j, - \int_{\alpha_1} \omega_j, \cdots, - \int_{\alpha_g} \omega_j \right) \quad (1 \leq j \leq g).
$$

Hence there are $b_1, ..., b_g \in \mathbf{C}$ such that

$$
\int_{\alpha_i} \psi - (2\pi\sqrt{-1}) l_i = \sum_{j=1}^{g} b_j \int_{\alpha_i} \omega_j, \quad \int_{\beta_i} \psi - (2\pi\sqrt{-1}) m_i = \sum_{j=1}^{g} b_j \int_{\beta_i} \omega_j,
$$

and then

$$
f = \exp \left(\int_{P_0}^{P} \left(\psi - \sum_{j=1}^{g} b_j \omega_j \right) \right)
$$

is a meromorphic function on R such that $\sum_{P \in R} \mathrm{ord}_P(f) \cdot P = D$. This implies the injectivity of μ.

Finally, we show that the surjectivity of μ. Let φ_1 be a nonzero holomorphic 1-form, and Q_1 be a point on R at which φ_1 does not vanish. Then by Riemann-Roch's theorem,

$$
\dim_{\mathbf{C}} H^0(R, \Omega_R(-Q_1)) = \dim_{\mathbf{C}} H^0(R, \mathcal{O}_R(Q_1)) + g - 2 = g - 1,
$$

and hence there are nonzero $\varphi_2 \in H^0(R, \Omega_R(-Q_1))$ and $Q_2 \in R$ at which φ_2 does not vanish. By repeating this process, one take a base $\varphi_1, ..., \varphi_g$ of $H^0(R, \Omega_R)$ and $Q_1, ..., Q_g \in R$ such that

$\varphi_{i+1}, ..., \varphi_g$, but not φ_i vanish at Q_i $(1 \leq i \leq g)$. Therefore, for $P_1, ..., P_g$ in neighborhoods of $Q_1, ..., Q_g$ respectively, the jacobian of

$$(P_1, ..., P_g) \longmapsto \left(\sum_{i=1}^{g} \int_{Q_i}^{P_i} \varphi_1, ..., \sum_{i=1}^{g} \int_{Q_i}^{P_i} \varphi_g \right)$$

is nonzero at $(Q_1, ..., Q_g)$, and hence by the implicit function theorem, the linear map μ is locally biholomorphic. This implies that this image $\mathrm{Im}(\mu)$ is an open subset of $\mathbf{C}^g/L$. By Riemann-Roch's theorem, for each divisor D with degree g on R, there is a nonzero element f of $H^0(R, \mathcal{O}_R(D))$, and hence

$$D + \sum_{P \in R} \mathrm{ord}_P(f) \cdot P$$

is a sum of g points on R. Therefore, for a fixed point $P_0 \in R$, the map

$$(P_1, ..., P_g) \mapsto \sum_{i=1}^{g} P_i - g \cdot P_0$$

gives a holomorphic surjection from R^g onto $\mathrm{Cl}^0(R)$. This implies that $\mathrm{Cl}^0(R)$ is compact, and hence $\mathrm{Im}(\mu)$ is closed in the connected set $\mathbf{C}^g/L$. Therefore, $\mathrm{Im}(\mu) = \mathrm{Cl}^0(R)$. QED.

Exercise 2. Fix $P_0 \in R$. Then for each $P \in R$, prove that there is a unique meromorphic 1-form $w_P = w_P(z)$ on R such that

- w_P is holomorphic except $z = P, P_0$;

- w_P has simple poles at $z = P, P_0$ with residues $1, -1$ respectively;

- $\displaystyle\int_{\alpha_i} w_P = 0$ $(1 \leq i \leq g)$.

Further, using the generalized Riemann's period relation, prove that

$$d \left(\int_{\beta_i} w_z \right) = 2\pi\sqrt{-1}\omega_i(z).$$

Example. If $R = \mathbf{C}/L$: genus 1; $L = \mathbf{Z} + \mathbf{Z}\tau$ $(\mathrm{Im}(\tau) > 0)$, then

$$H^0(R, \Omega_R) = \mathbf{C}dz, \quad Z = \int_0^{\tau} dz = \tau.$$

§3. Schottky uniformization

3.1. Degeneration of Riemann surfaces

__Genus 1 case.__ If $f(x)$ (: degree 3, without multiple root) tends to $a(x-\alpha)^2(x-\beta)$ ($a \neq 0, \alpha \neq \beta$), then the complex torus $C_f(\mathbf{C})$ degenerates to a singular space obtained by identifying 2-points on the Riemann sphere (__Figure__).

For example, for $f(x) = (x^2 - \varepsilon^2)(x + 1)$,

$$y^2 = f(x) \Leftrightarrow (\sqrt{x^2 + x^3} + y)(\sqrt{x^2 + x^3} - y) = \varepsilon^2(1 + x)$$
$$\overset{\varepsilon \to 0}{\longrightarrow} \ (\sqrt{x^2 + x^3} + y)(\sqrt{x^2 + x^3} - y) = 0 \ \text{ around } (x, y) = (0, 0),$$

where $\sqrt{x^2 + x^3} = \sum_{k=0}^{\infty} \dbinom{1/2}{k} x^{k+1}$.

__Local degeneration.__ For a complex number ε such that $0 < |\varepsilon| < 1$, let D be the union of the two annular domains:

$$U = \{x \in \mathbf{C} \mid |\varepsilon| < |x| < 1\}, \quad V = \{y \in \mathbf{C} \mid |\varepsilon| < |y| < 1\}$$

by the relation $xy = \varepsilon$. Then under $\varepsilon \to 0$, D becomes the union of the 2 disks

$$\{x \in \mathbf{C} \mid |x| < 1\}, \quad \{y \in \mathbf{C} \mid |y| < 1\}$$

identifying $x = 0$ and $y = 0$.

__Ordinary double points.__ For a point P on a curve C,

$$P \text{ is an } \textbf{ordinary double point} \text{ (or } \textbf{node})$$
$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{the local equation around } P \in C \text{ is given by } xy = 0 \\ \text{for some formal coordinates } x, y \end{cases}$$
$$\Longleftrightarrow \ P \text{ is a point of multiplicity 2 with distinct tangent directions}$$

3.2. Schottky uniformization of Riemann surfaces

__Schottky uniformization__ is to construct Riemann surfaces of genus g from a $2g$ holed Riemann sphere by identifying these holes in pairs (__Figure__). More precisely, let

$$PGL_2(\mathbf{C}) \overset{\text{def}}{=} GL_2(\mathbf{C})/\mathbf{C}^\times(\cdot E_2)$$

which acts on $\mathbb{P}^1(\mathbf{C})$ by the Möbius transformation, and let

$D_{\pm 1}, ..., D_{\pm g} \subset \mathbb{P}^1(\mathbf{C})$: disjoint closed domains bounded by Jordan curves ∂D_i,

$\gamma_1, ..., \gamma_g \in PGL_2(\mathbf{C})$ such that $\gamma_i(\mathbb{P}^1(\mathbf{C}) - D_{-i}) = $ the interior D_i° of D_i,

$\Gamma \overset{\text{def}}{=} \langle \gamma_1, ..., \gamma_g \rangle$: the subgroup of $PGL_2(\mathbf{C})$ generated by $\gamma_1, ..., \gamma_g$,

$\Omega_\Gamma \overset{\text{def}}{=} \bigcup_{\gamma \in \Gamma} \gamma \left(\mathbb{P}^1(\mathbf{C}) - \bigcup_{i=1}^{g} (D_i^\circ \cup D_{-i}^\circ) \right).$

Then the Riemann surface

$$R_\Gamma \overset{\text{def}}{=} \left(\mathbb{P}^1(\mathbf{C}) - \bigcup_{i=1}^{g}(D_i^\circ \cup D_{-i}^\circ)\right) \Big/ \partial D_i \overset{\gamma_i}{\approx} \partial D_{-i} \ (: \text{gluing by } \gamma_i)$$

$$= \ \Omega_\Gamma/\Gamma$$

is called (Schottky) uniformized by the **Schottky group** Γ. It is known as Koebe's theorem that any Riemann surface can be Schottky uniformized. Counterclockwise oriented loops ∂D_i and oriented paths from $w_i \in \partial D_{-i}$ to $\gamma_i(w_i) \in \partial D_i$ $(1 \le i \le g)$ become canonical generators, and we denote them by α_i, β_i respectively (**Figure**).

Remark. Denote by $\mathbb{H}^3$ the 3-dimensional hyperbolic space. Then the quotient hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ becomes a handlebody whose boundary is R_Γ.

Exercise 3. Prove that Γ is a free group with generators $\gamma_1, ..., \gamma_g$, and that the action of Γ on Ω_Γ is free and properly discontinuous. Further, prove that each γ_i $(1 \le i \le g)$ is uniquely represented by

$$\gamma_i = \begin{pmatrix} t_i & t_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & s_i \end{pmatrix} \begin{pmatrix} t_i & t_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \bmod(\mathbf{C}^\times),$$

where $t_i \in D_i^\circ$, $t_{-i} \in D_{-i}^\circ$ and $|s_i| < 1$ (hence γ_i is hyperbolic (or loxodromic)), and that

$$t_{\pm i} \ = \ \lim_{n\to\infty} \gamma_i^{\pm n}(z) \ (z \in \Omega_\Gamma).$$

t_i, t_{-i} are called the **attractive**, **repulsive** fixed point of γ_i respectively, and s_i is called the **multiplier** of γ_i.

3.3. Explicit formula of periods

Theorem 3.1. (Schottky [S]) *Assume that $\infty \in \Omega_\Gamma$ and that $\sum_{\gamma\in\Gamma} |\gamma'(z)|$ converges uniformly on any compact subset of*

$$\Omega_\Gamma - \bigcup_{\gamma\in\Gamma} \gamma(\infty).$$

Then we have
(1) For $n \ge 1$ and a point $p \in \Omega_\Gamma - \bigcup_{\gamma\in\Gamma} \gamma(\infty)$,

$$w_{n,p}(z) \overset{\text{def}}{=} \sum_{\gamma\in\Gamma} \frac{d\gamma(z)}{(\gamma(z)-p)^n} = \sum_{\gamma\in\Gamma} \frac{\gamma'(z)}{(\gamma(z)-p)^n}dz$$

becomes a meromorphic 1-form on R_Γ. If $n > 1$, then $w_{n,p}$ is of the 2-nd kind, and it has only poles (of order n) at the point $\overline{p}$ on R_Γ induced from p. If $n = 1$, then $w_{n,p}$ is of the 3-rd kind, and it has only simple poles at $\overline{p}, \overline{\infty}$ with residues $1, -1$ respectively. Furthermore, for $n \ge 0$,

$$\sum_{\gamma\in\Gamma} \gamma(z)^n d\gamma(z) = \sum_{\gamma\in\Gamma} \gamma(z)^n \cdot \gamma'(z)dz$$

becomes a meromorphic 1-form on R_Γ which has only pole (of order $n+2$) at $\overline{\infty}$.

(2) *For $i = 1, ..., g$,*

$$\omega_i(z) = \frac{1}{2\pi\sqrt{-1}} \sum_{\gamma \in \Gamma/\langle\gamma_i\rangle} \left(\frac{1}{z - \gamma(t_i)} - \frac{1}{z - \gamma(t_{-i})} \right) dz$$

give a basis of $H^0(R_\Gamma, \Omega_{R_\Gamma})$ satisfying that $\int_{\alpha_i} \omega_j = \delta_{ij}$.

(3) *For $1 \le i, j \le g$ and $\gamma \in \Gamma$, put*

$$\psi_{ij}(\gamma) = \begin{cases} s_i & (if \ i = j \ and \ \gamma \in \langle\gamma_i\rangle), \\ \dfrac{(t_i - \gamma(t_j))(t_{-i} - \gamma(t_{-j}))}{(t_i - \gamma(t_{-j}))(t_{-i} - \gamma(t_j))} & (otherwise), \end{cases}$$

where t_i, t_{-i} are the attractive, repulsive fixed points of γ_i respectively, and s_i is the multiplier of γ_i. Then we have

$$\exp\left(2\pi\sqrt{-1}z_{ij}\right) = \prod_{\gamma \in \langle\gamma_i\rangle \backslash \Gamma / \langle\gamma_j\rangle} \psi_{ij}(\gamma),$$

where $Z = (z_{ij})_{i,j}$ is the period matrix of $(R_\Gamma; (\alpha_i, \beta_i)_{1 \le i \le g})$.

Proof. The assertion (1) is evident except the convergence of $w_{n,p}(z)$ which follows from the assumption and that the action of Γ on Ω_Γ is properly discontinuous. Further, $w_{1,p}(z)$ has simple poles at $\overline{p}, \overline{\infty}$ with residues $1, -1$ respectively, and satisfies that $\int_{\alpha_i} w_{1,p} = 0$ ($1 \le i \le g$). Then by Exercise 2,

$$
\begin{aligned}
2\pi\sqrt{-1}\omega_i(z) &= d\left(\int_{\zeta_i}^{\gamma_i(\zeta_i)} \sum_{\gamma \in \Gamma} \frac{d\gamma(\zeta)}{\gamma(\zeta) - z} \right) \ ; \ \zeta_i \text{ is a point on } \partial D_{-i} \\
&= d\left(\sum_{\gamma \in \Gamma} \log\left(\frac{(\gamma\gamma_i)(\zeta_i) - z}{\gamma(\zeta_i) - z} \right) \right) \\
&= \sum_{\gamma \in \Gamma} \left(\frac{1}{z - (\gamma\gamma_i)(\zeta_i)} - \frac{1}{z - \gamma(\zeta_i)} \right) dz \\
&= \sum_{\gamma \in \Gamma/\langle\gamma_i\rangle} \sum_{n \in \mathbf{Z}} \left(\frac{1}{z - (\gamma\gamma_i^{n+1})(\zeta_i)} - \frac{1}{z - (\gamma\gamma_i^n)(\zeta_i)} \right) dz,
\end{aligned}
$$

and since $t_{\pm i} = \lim_{n \to \infty} \gamma_i^{\pm n}(w_i) \in D_{\pm i}^{\circ}$ (Exercise 3), we have

$$\omega_i(z) = \frac{1}{2\pi\sqrt{-1}} \sum_{\gamma \in \Gamma/\langle\gamma_i\rangle} \left(\frac{1}{z - \gamma(t_i)} - \frac{1}{z - \gamma(t_{-i})} \right) dz,$$

which proves (2). QED.

Exercise 4. Prove that $\int_{\alpha_i} w_{1,p} = 0$ ($1 \le i \le g$), and check that ω_i is Γ-invariant and $\int_{\alpha_i} \omega_j = \delta_{ij}$.

Exercise 5. Prove (3) of Theorem 3.1.

Proposition 3.2. *Assume that $\Omega_\Gamma \ni \infty$, and that $t_{\pm i}$ are fixed and s_i are sufficiently small, then the assumption in Theorem 3.1 is satisfied.*

Proof. For 2 disks $D_i, D_j \subset \mathbf{C}$ with radius r_i, r_j respectively, put

$$
\begin{aligned}
\rho_{i,j} \quad &: \quad \text{the distance between the centers of } D_i \text{ and } D_j, \\
K_{i,j} \quad &= \quad \frac{(r_i^2 + r_j^2 - \rho_{i,j}^2)^2}{4r_i^2 r_j^2} - 1 \geq 0, \\
L_{i,j} \quad &= \quad \frac{1}{\sqrt{1 + K_{i,j}} + \sqrt{K_{i,j}}} \leq 1.
\end{aligned}
$$

Then $K_{i,j}$ and $L_{i,j}$ are invariant under any Möbius transformation, and $r_i \leq L_{i,j} \cdot r_j$ if $D_i \subset D_j$. Under the assumption, one can take disks $D_{\pm 1}, ..., D_{\pm g}$ such that the sum of $L_{i,j}$ $(i, j \in \{\pm 1, ..., \pm g\}, i \neq j)$ is smaller than 1. Hence by the above, there is a positive constant C such that if $\gamma = \prod_{s=1}^{l} \gamma_{k(s)} \in \Gamma$ is expressed as

$$
\begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \mod(\mathbf{C}^\times); \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in SL_2(\mathbf{C}),
$$

then

$$
\frac{1}{|c_\gamma|^2} \leq C \cdot \prod_{s=1}^{l-1} L_{-k(s), k(s+1)}.
$$

Therefore,

$$
\sum_{\gamma \in \Gamma - \{1\}} \frac{1}{|c_\gamma|^2} \leq C \cdot \sum_{m=0}^{\infty} \left(\sum_{i \neq j} L_{i,j} \right)^m < \infty,
$$

and hence

$$
\sum_{\gamma \in \Gamma} |\gamma'(z)| \leq 1 + \frac{1}{d(z)^2} \sum_{\gamma \in \Gamma - \{1\}} \frac{1}{|c_\gamma|^2}
$$

satisfies the condition since $d(z) \stackrel{\text{def}}{=} \min\{|z - \gamma^{-1}(\infty)|; \gamma \in \Gamma\} > 0$ is bounded on any compact subset outside $\bigcup_{\gamma \in \Gamma} \gamma(\infty)$. QED

Remark. Schottky [S] gives a (more geometric) convergence condition on $\sum_{\gamma \in \Gamma} |\gamma'(z)|$ as follows: all $\partial D_{\pm i}$ can be taken as circles (in this case, Γ is called classical) and there are $2g - 3$ circles $C_1, ..., C_{2g-3}$ in $F = \mathbb{P}^1(\mathbf{C}) - \bigcup_{i=1}^{g} \left(D_i^\circ \cup D_{-i}^\circ \right)$ satisfying that

- $C_1, ..., C_{2g-3}, \partial D_{\pm 1}, ..., \partial D_{\pm g}$ are mutually disjoint;

- $C_1, ..., C_{2g-3}$ divide F into $2g - 2$ domains $R_1, ..., R_{2g-2}$;

- each R_i has exactly three boundary circles.

Variation of forms and periods. Let $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$ be a Schottky group of rank g as above, and put $\Gamma' = \langle \gamma_1, ..., \gamma_{g-1} \rangle$ which is a Schottky group of rank $g-1$. If the multiplier

$$s_g = \frac{\gamma_g(z) - t_g}{z - t_g} \cdot \frac{z - t_{-g}}{\gamma_g(z) - t_{-g}}$$

$: \quad$ the product of local coordinates around t_g, t_{-g} respectively

of γ_g tends to 0, then

- $R_\Gamma \longrightarrow \begin{cases} \text{the singular curve } \widetilde{R}_{\Gamma'} \text{ with unique singular (ordinary double) point} \\ \text{obtained from } R_{\Gamma'} \text{ by identifying } t_g \text{ and } t_{-g}; \end{cases}$

- $2\pi\sqrt{-1}\,\omega_i(z) = \displaystyle\sum_{\gamma \in \Gamma/\langle \gamma_i \rangle} \left(\frac{1}{z - \gamma(t_i)} - \frac{1}{z - \gamma(t_{-i})} \right) dz \in H^0\left(R_\Gamma, \Omega_{R_\Gamma} \right)$

$$\longrightarrow \begin{cases} \displaystyle\sum_{\gamma \in \Gamma'/\langle \gamma_i \rangle} \left(\frac{1}{z - \gamma(t_i)} - \frac{1}{z - \gamma(t_{-i})} \right) dz & (i < g), \\[6pt] \left(\dfrac{1}{z - t_g} - \dfrac{1}{z - t_{-g}} \right) dz + \cdots & (i = g) \end{cases}$$

   which has a pole at the ordinary double point $t_g = t_{-g}$ on $\widetilde{R}_{\Gamma'}$ if $i = g$;

- (Fay's formula [Fay]) $p_{ij} \longrightarrow \begin{cases} \text{the multiplicative periods of } R_{\Gamma'} & (i, j < g), \\ 0 & (i = j = g). \end{cases}$

Therefore, on the complex geometry of $\widetilde{R}_{\Gamma'}$, it is natural to replace the sheaf of holomorphic 1-forms on $\widetilde{R}_{\Gamma'}$ by that of 1-forms η on $R_{\Gamma'}$ holomorphic except for simple poles at t_g, t_{-g} satisfying that $\mathrm{Res}_{t_g}(\eta) + \mathrm{Res}_{t_{-g}}(\eta) = 0$ (see 5.2 below).

Remark. We can obtain variational formula under other degenerations (see [I3]).

3.4. Fractal nature of Schottky groups (cf. [MumSW])

Limit set. The **limit set** L_Γ of Γ is defined to be the complement of Ω_Γ in $\mathbb{P}^1(\mathbf{C})$. When we take limits of Schottky groups such as the above domains $D_{\pm i} \subset \mathbb{P}^1(\mathbf{C})$ are tangent, their limit sets become *fractal* pictures. These limiting process is also important in the study of degeneration of R_Γ and deformation of hyperbolic 3-manifolds.

   Let Γ be a Kleinian group generated by $\gamma_1, \gamma_2 \in PGL_2(\mathbf{C})$ such that there are tangential disks $D_{\pm 1}, D_{\pm 2} \subset \mathbb{P}^1(\mathbf{C})$ satisfying

$$\gamma_i \left(\mathbb{P}^1(\mathbf{C}) - D_{-i} \right) = D_i^\circ, \ \gamma_i(\{\text{tangential points}\}) = \{\text{tangential points}\} \ (i = 1, 2),$$

and consider the following cases.

Case 1. $D_{\pm 1} \cup D_{\pm 2} \subset \mathbb{P}^1(\mathbf{C})$ is *homeomorphic to*

$$\left\{ z \in \mathbf{C}; |z - \pm(1 + \sqrt{-1})| \leq 1 \right\} \cup \left\{ z \in \mathbf{C}; |z - \pm(1 - \sqrt{-1})| \leq 1 \right\}.$$

Then Γ is called a once-punctured torus group, and R_Γ becomes a union of two tori at one point (cf. [MumSW, p.189–190]).

<u>Case 2.</u> Each of $D_{\pm 1}, D_{\pm 2} \subset \mathbb{P}^1(\mathbf{C})$ is tangent to other three disks. Then L_Γ becomes an Apollonian gasket, and R_Γ becomes a union of two degenerate tori at one point (cf. [MumSW, p.205]).

<u>Case 3.</u> $D_{\pm 1} \cup D_{\pm 2} \subset \mathbb{P}^1(\mathbf{C})$ is *homeomorphic to*

$$\left\{ z \in \mathbf{C}; |z - (\pm 1 + \sqrt{-1})| \leq 1 \right\} \cup \left\{ z \in \mathbf{C}; |z - (\pm 1 - \sqrt{-1})| \leq 1 \right\}.$$

Then R_Γ becomes a union of two spheres at three points (cf. [MumSW, p.216]).

§4. Arithmetic uniformization

4.1. Periods as power series

First, we calculate the periods p_{ij} given in Theorem 3.1 (3) (cf. Exercise 5) as power series over $\mathbf{Z}$ by regarding the fixed points and multiplier $t_{\pm i}, s_i$ of γ_i as variables $x_{\pm i}, y_i$ respectively. Let Γ_Δ be a subgroup of PGL_2 generated by

$$\phi_i \stackrel{\text{def}}{=} \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y_i \end{pmatrix} \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \mod(\mathbf{G}_m) \quad (1 \le i \le g).$$

Put

$$A_0 = \mathbf{Z}\left[\frac{(x_i - x_j)(x_k - x_l)}{(x_i - x_l)(x_k - x_j)} \left(\begin{array}{ll} i,j,k,l & \in \{\pm 1, ..., \pm g\} \\ & : \text{ mutually different} \end{array} \right) \right],$$

$$A_\Delta = A_0[[y_1, ..., y_g]],$$

and let I_Δ be the ideal of A_Δ generated by $y_1, ..., y_g$. By definition,

$$p_{ij} = \prod_{\phi \in \langle \phi_i \rangle \backslash \Gamma_\Delta / \langle \phi_j \rangle} \psi_{ij}(\phi),$$

where

$$\psi_{ij}(\phi) = \begin{cases} y_i & (\text{if } i = j \text{ and } \phi \in \langle \phi_i \rangle), \\ \dfrac{(x_i - \phi(x_j))(x_{-i} - \phi(x_{-j}))}{(x_i - \phi(x_{-j}))(x_{-i} - \phi(x_j))} & (\text{otherwise}). \end{cases}$$

Put $\phi_{-i} \stackrel{\text{def}}{=} \phi_i^{-1}$ $(1 \le i \le g)$. Then

$$\Phi_{ij} = \left\{ \phi = \phi_{\sigma(1)} \cdots \phi_{\sigma(n)} \left| \begin{array}{l} \sigma(1) \ne \pm i, \ \sigma(n) \ne \pm j, \\ \sigma(k) \ne -\sigma(k+1) \ (1 \le k \le n-1) \end{array} \right. \right\}$$

gives a set of complete representatives of $\langle \phi_i \rangle \backslash \Gamma_\Delta / \langle \phi_j \rangle$. If $\alpha \in x_j + I_\Delta$ with $j \ne -i$, then

$$\phi_i(\alpha) = \left(x_i - \frac{(\alpha - x_i)x_{-i}y_i}{\alpha - x_{-i}} \right) \left(1 - \frac{(\alpha - x_i)y_i}{\alpha - x_{-i}} \right)^{-1} \in x_i + I_\Delta.$$

Hence if $\phi = \phi_{\sigma(1)} \cdots \phi_{\sigma(n)} \in \Phi_{ij}$, then $\phi(x_{\pm j}) \in x_{\sigma(1)} + I_\Delta$ and

$$
\begin{aligned}
& \phi(x_j) - \phi(x_{-j}) \\
&= \frac{(x_{\sigma(1)} - x_{-\sigma(1)})^2 (\phi'(x_j) - \phi'(x_{-j}))y_{\sigma(1)}}{(\phi'(x_j) - x_{-\sigma(1)} - y_{\sigma(1)}(\phi'(x_j) - x_{\sigma(1)}))(\phi'(x_{-j}) - x_{-\sigma(1)} - y_{\sigma(1)}(\phi'(x_{-j}) - x_{\sigma(1)}))} \\
& \quad (\phi' \stackrel{\text{def}}{=} \phi_{\sigma(2)} \cdots \phi_{\sigma(n)}) \\
&= \cdots \in I_\Delta^n.
\end{aligned}
$$

by inductive calculus. Therefore,

$$\frac{(x_i - \phi(x_j))(x_{-i} - \phi(x_{-j}))}{(x_i - \phi(x_{-j}))(x_{-i} - \phi(x_j))} = 1 + \frac{(x_i - x_{-i})(\phi(x_j) - \phi(x_{-j}))}{(x_i - \phi(x_{-j}))(x_{-i} - \phi(x_j))} \in 1 + I_\Delta^n,$$

and hence p_{ij} are elements of A_Δ calculated as

$$p_{ij} \;=\; c_{ij}\left(1 + \sum_{|k|\neq i,j} \frac{(x_i - x_{-i})(x_j - x_{-j})(x_k - x_{-k})^2}{(x_i - x_k)(x_{-i} - x_k)(x_j - x_{-k})(x_{-j} - x_{-k})} y_{|k|} + \cdots\right),$$

where

$$c_{ij} \;\overset{\mathrm{def}}{=}\; \begin{cases} y_i & (\text{if } i = j), \\[2mm] \dfrac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_i - x_{-j})(x_{-i} - x_j)} & (\text{if } i \neq j). \end{cases}$$

4.2. Tate curve and Mumford curves

In order to study geometric meaning of the above calculation, we review the theory of the Tate curve and its higher genus version called arithmetic uniformization theory.

Tate curve. Recall that an elliptic curve $\mathbf{C}/L$ is defined by the equation (see 2.1):

$$y^2 = 4x^3 - 60E_4(L)x - 140E_6(L).$$

Therefore, if

$$\begin{aligned} x &= (2\pi\sqrt{-1})^2\left(X + \frac{1}{12}\right), & y &= (2\pi\sqrt{-1})^3\,(2Y + X), \\[2mm] a_4 &= -\frac{15E_4(L)}{(2\pi\sqrt{-1})^4} + \frac{1}{48}, & a_6 &= -\frac{35E_6(L)}{(2\pi\sqrt{-1})^6} - \frac{5E_4(L)}{4(2\pi\sqrt{-1})^4} + \frac{1}{1728}, \end{aligned}$$

then the above equation is equivalent to

$$Y^2 + XY = X^3 + a_4 X + a_6.$$

Furthermore, if $L = \mathbf{Z} + \mathbf{Z}\tau$ and $q = e^{2\pi\sqrt{-1}\tau}$, then by the calculation of the **Eisenstein series** (see Exercise 6 below):

$$\sum_{u\in L-\{0\}} \frac{1}{u^{2k}} = 2\zeta(2k) + \frac{2(2\pi\sqrt{-1})^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\,q^n \quad (k > 1),$$

where

$$\zeta(2k) \overset{\mathrm{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} : \text{ the \textbf{zeta values}, and } \sigma_{2k-1}(n) \overset{\mathrm{def}}{=} \sum_{d|n} d^{2k-1},$$

we have

$$\begin{aligned} a_4(q) &= -5\sum_{n=1}^{\infty} \sigma_3(n)\,q^n = -5q - 45q^2 + \cdots, \\[2mm] a_6(q) &= -\frac{1}{12}\sum_{n=1}^{\infty}(5\sigma_3(n) + 7\sigma_5(n))\,q^n = -q - 23q^2 + \cdots. \end{aligned}$$

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Exercise 6. Prove that

$$\zeta(2k) = -\frac{(2\pi\sqrt{-1})^{2k}}{2(2k)!}B_{2k}$$

$$\left(B_n \text{ is the } n\text{-th } \textbf{Bernoulli numbers} \text{ given by } \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\right)$$

$$\Rightarrow \quad \zeta(2) = \frac{\pi^2}{6}, \ \zeta(4) = \frac{\pi^4}{90}, \ \zeta(6) = \frac{\pi^6}{945},$$

and

$$\sum_{(m,n)\in\mathbf{Z}^2-\{(0,0)\}} \frac{1}{(m+n\tau)^{2k}} = 2\zeta(2k) + \frac{2(2\pi\sqrt{-1})^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) \ q^n \quad (k > 1),$$

from the well-known formula:

$$\pi \cot(\pi a) = \frac{1}{a} + \sum_{m=1}^{\infty} \left(\frac{1}{a+m} + \frac{1}{a-m}\right) \ \left(\Leftrightarrow \ \sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)\right)$$

by substituting x to $2\pi\sqrt{-1}a$, and differentiating the formula successively and substituting $n\tau$ to a respectively.

Exercise 7. Show that $a_4(q)$ and $a_6(q)$ belong to the ring

$$\mathbf{Z}[[q]] \overset{\text{def}}{=} \left\{\sum_{n=0}^{\infty} c_n q^n \ \middle| \ c_n \in \mathbf{Z}\right\}$$

of formal power series of q with coefficients in $\mathbf{Z}$.

The **Tate curve** is the curve over $\mathbf{Z}[[q]]$ defined by

$$y^2 + xy = x^3 + a_4(q)x + a_6(q).$$

Then Tate proved the following:

Theorem 4.1. ([Si, T])
  (1) *The Tate curve becomes an elliptic curve over the ring*

$$\mathbf{Z}((q)) \overset{\text{def}}{=} \mathbf{Z}[[q]] [1/q] = \left\{\sum_{n>m}^{\infty} c_n q^n \ \middle| \ m \in \mathbf{Z}, \ c_n \in \mathbf{Z}\right\}$$

of Laurent power series of q with coefficients in $\mathbf{Z}$.
  (2) *Recall $\sigma_1(n) = \sum_{d|n} d$, and put*

$$X(u,q) = \sum_{n\in\mathbf{Z}} \frac{q^n u}{(1-q^n u)^2} - 2\sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

$$Y(u,q) = \sum_{n\in\mathbf{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + \sum_{n=1}^{\infty} \sigma_1(n)q^n.$$

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Then $z \mapsto \left(X(e^{2\pi\sqrt{-1}z}, e^{2\pi\sqrt{-1}\tau}), Y(e^{2\pi\sqrt{-1}z}, e^{2\pi\sqrt{-1}\tau}) \right)$ *gives rise to an isomorphism between* $\mathbf{C}/L$ *and the elliptic curve* E_τ *over* $\mathbf{C}$ *obtained from the Tate curve by substituting* $q = e^{2\pi\sqrt{-1}\tau}$.

(3) *Let K be a complete valuation field with multiplicative valuation $|\cdot|$, and let $q \in K^\times$ such that $|q| < 1$. Then by the substitution the variable $q \mapsto q \in K^\times$, the series $a_4(q)$ and $a_6(q)$ converge in K, and the Tate curve gives an elliptic curve E_q over K. Further, we have an isomorphism:*

$$K^\times/\langle q \rangle \xrightarrow{\sim} E_q(K)$$

$$u \bmod \langle q \rangle \longmapsto \begin{cases} (X(u,q), Y(u,q)) & (u \notin \langle q \rangle), \\ 0 & (u \in \langle q \rangle). \end{cases}$$

Proof. (1) The discriminant Δ of the Tate curve is given by

$$-a_6(q) + a_4(q)^2 + 72a_4(q)a_6(q) - 64a_4(q)^3 - 432a_6(q)^2$$

$$= \quad q - 24q^2 + \cdots : \text{ a formal power series with integral coefficients}$$

$$\overset{\text{in fact}}{=} \quad q \prod_{n=1}^{\infty} (1-q^n)^{24} : \text{ a cusp form of weight 12 for } SL_2(\mathbf{Z}).$$

Therefore, the Tate curve is smooth over $\mathbf{Z}[[q]]\,[1/\Delta] = \mathbf{Z}((q))$.

(2) First, note that the following hold:

$$\frac{\wp_L(z)}{(2\pi\sqrt{-1})^2} = \sum_{n\in\mathbf{Z}} \frac{q^n u}{(1-q^n u)^2} + \frac{1}{12} - 2s_1(q) \quad \left(s_1(q) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \sigma_1(n)q^n \right),$$

$$\frac{\wp_L'(z)}{(2\pi\sqrt{-1})^3} = \sum_{n\in\mathbf{Z}} \frac{q^n u(1 + q^n u)}{(1-q^n u)^3}.$$

Because the right hand sides are q-series which are invariant under $u \mapsto qu$, hence invariant under $z \mapsto z+1, z+\tau$, and they have the expansions of z as

$$\frac{u}{(1-u)^2} + \frac{1}{12} + \sum_{n\neq 0} \frac{q^n}{(1-q^n)^2} - 2s_1(q) + O(z) = \frac{1}{\left(2\pi\sqrt{-1}z\right)^2} + O(z),$$

$$\frac{u(1+u)}{(1-u)^3} + \sum_{n\neq 0} \frac{q^n(1+q^n)}{(1-q^n)^3} + O(z) = \frac{-2}{\left(2\pi\sqrt{-1}z\right)^3} + O(z)$$

which are equal to those of the left hand sides respectively. Therefore,

$$X = \frac{x}{(2\pi\sqrt{-1})^2} - \frac{1}{12} = \frac{\wp_L(z)}{(2\pi\sqrt{-1})^2} - \frac{1}{12} = X(u,q),$$

$$Y = \frac{y}{2(2\pi\sqrt{-1})^3} - \frac{x}{2(2\pi\sqrt{-1})^2} + \frac{1}{24}$$

$$= \frac{\wp_L'(z)}{2(2\pi\sqrt{-1})^3} - \frac{\wp_L(z)}{2(2\pi\sqrt{-1})^2} + \frac{1}{24}$$

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$$= \frac{1}{2} \sum_{n \in \mathbf{Z}} \frac{q^n u (1 + q^n u)}{(1 - q^n u)^3} - \frac{1}{2} \sum_{n \in \mathbf{Z}} \frac{q^n u}{(1 - q^n u)^2} + s_1(q)$$

$$= Y(u, q).$$

As seen in 2.1, $z + L \mapsto (x = \wp_L(z), y = \wp'_L(z))$ is an isomorphism from $\mathbf{C}/L$ onto the elliptic curve $y^2 = 4x^3 - 60E_4(L)x - 140E_6(L)$, and hence

$$z + L \mapsto (X = X(u, q), Y = Y(u, q))$$

gives an isomorphism $\mathbf{C}/L \stackrel{\sim}{\to} E_\tau$.

(3) By substituting the variable $q \mapsto q \in K^\times$ with $|q| < 1$, $\Delta = q - 24q^2 + \cdots$ satisfies that $|\Delta| = |q| \neq 0$, and hence E_q is an elliptic curve over K. By (2), $X(u, q)$ and $Y(u, q)$ satisfies the equation of the Tate curve:

$$Y(u, q)^2 + X(u, q)Y(u, q) = X(u, q)^3 + a_4(q)X(u, q) + a_6(q)$$

for all complex numbers u, q in a certain convergence domain, and hence this equation holds as formal power series in q with coefficients in $\mathbf{Q}(u)$. Therefore, by substituting the variable $q \mapsto q \in K^\times$ with $|q| < 1$, one can see that the map in (3) is well-defined, and is evidently injective. The addition law on the Tate curve is given by

$$P_i = (x_i, y_i) \; (i = 1, 2, 3), \quad P_1 + P_2 = P_3$$
$$\longrightarrow \begin{cases} (x_2 - x_1)^2 x_3 = (y_2 - y_1)^2 + (y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)^2(x_1 + x_2), \\ (x_2 - x_1)y_3 = (-(y_2 - y_1) + (x_2 - x_1))x_3 - (y_1 x_2 - y_2 x_1), \end{cases}$$

if $x_1 \neq x_2$. Hence by (2), this holds if $x_i = X(u_i, q)$, $y_i = X(u_i, q)$, $(i = 1, 2, 3)$ with $u_1 u_2 = u_3$ for all complex numbers u_1, u_2, q in a certain convergence domain, and hence holds as formal power series in q with coefficients in $\mathbf{Q}(u_1, u_2)$. Therefore, by substituting the variable $q \mapsto q \in K^\times$ with $|q| < 1$, one can see that the map in (3) is a homomorphism. We omit the surjectivity of the map which is most hardest part of the proof. QED.

Remark. Similar argument to the proof of Theorem 4.1 (3) is used in [I1] to show that p-adic theta functions of Mumford curves give solutions to soliton equations.

Mumford curves. Mumford [Mu2] gave a higher genus version of the Tate curve over complete local domains as an analogy of Schottky uniformization theory, i.e., for a complete integrally closed noetherian local ring R with quotient field K, and a Schottky group $\Gamma \subset PGL_2(K)$ over K which is *flat* over R, he constructed a **Mumford curve** over $(R \subset)K$ which is a proper smooth curve C_Γ over K obtained as the general fiber of a stable curve over R uniformized by Γ such that its special fiber consists of (may be singular) projective lines and its singularities are all k-rational (k is the residue field of R). Furthermore, he showed that $\Gamma \mapsto C_\Gamma$ gives rise to the following bijection:

$$\left\{ \begin{array}{l} \text{Conjugacy classes of flat} \\ \text{Schottky groups over } (R \subset)K \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{Mumford curves over } (R \subset)K \end{array} \right\}$$

If K is a complete valuation field, then any Schottky group Γ over K is flat over its valuation ring, and it is shown in [GP] that C_Γ is given as the quotient by Γ of its region of discontinuity in $K \cup \{\infty\}$ (important examples of rigid analytic geometry).

4.3. Arithmetic Schottky uniformization

Stable curves. A **stable curve** of genus $g > 1$ over a scheme S is a proper and flat morphism $C \to S$ whose geometric fibers are reduced and connected 1-dimensional schemes C_s such that

- C_s has only ordinary double points;

- $\mathrm{Aut}(C_s)$ is a finite group, i.e., if X is a smooth rational component of C_s, then X meets the other components of C_s at least 3 points;

- the dimension of $H^1(C_s, \mathcal{O}_{C_s})$ is equal to g.

Degenerate curves and dual graphs. A **degenerate curve** is a stable curve whose irreducible components are (may be singular) projective lines. For a degenerate curve, by the correspondence:

$$\begin{aligned} \text{its irreducible components} &\longleftrightarrow \text{vertices} \\ \text{its singular points} &\longleftrightarrow \text{edges} \end{aligned}$$

(an irreducible component contains a singular point if and only if the corresponding vertex is contained in (or adjacent to) the corresponding edge), we have its **dual graph** which becomes a stable graph, i.e., a connected and finite graph whose vertices has at least 3 branches (**Figure**). For a degenerate curve C with dual graph Δ,

$$\begin{aligned} \text{the genus of } C &= \mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z}) \\ &= \text{the number of generators of the free group } \pi_1(\Delta). \end{aligned}$$

Since any triplet of distinct points on $\mathbb{P}^1$ is uniquely translated to $(0, 1, \infty)$ by the action of PGL_2, for a stable graph Δ, the moduli space of degenerate curves with dual graph Δ has dimension

$$\sum_{v: \text{ vertices of } \Delta} (\deg(v) - 3),$$

where $\deg(v)$ denotes the number of branches ($\neq$ edges) starting from v. In particular, a stable graph is **trivalent**, i.e., all the vertices have just 3 branches if and only if the corresponding curves are **maximally degenerate** which means that this moduli consists of only one point.

Exercise 8. For any stable graph Δ, prove that

$$\sum_{v: \text{ vertices of } \Delta} (\deg(v) - 3) + \text{the number of edges of } \Delta = 3 \left(\mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z}) - 1 \right).$$

General degenerating process. (Ihara and Nakamura [IhN]). For a stable graph Δ with orientation on each edge,

$$\begin{aligned} g &\overset{\mathrm{def}}{=} \mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z}), \\ P_v &\overset{\mathrm{def}}{=} \mathbb{P}^1(\mathbf{C}) \quad (v : \text{vertices of } \Delta). \end{aligned}$$

and for each oriented edge e ($v_{-e} \overset{e}{\to} v_e$) of Δ, let

$$v_e \overset{\text{def}}{=} \text{ the end point of } e,$$
$$v_{-e} \overset{\text{def}}{=} \text{ the starting point of } e,$$
$$\gamma_e \quad : \quad \text{a hyperbolic element of } PGL_2(\mathbf{C}) \text{ which gives } \gamma_e : P_{v_{-e}} \overset{\sim}{\to} P_{v_e},$$
$$t_e \quad \in \quad P_{v_e} : \text{the attractive fixed point of } \gamma_e,$$
$$t_{-e} \quad \in \quad P_{v_{-e}} : \text{the repulsive fixed point of } \gamma_e.$$

Fix a vertex v_0 of Δ, and put

$$\Gamma \overset{\text{def}}{=} \left\{ \gamma_{e_1}^{i_1} \cdots \gamma_{e_n}^{i_n} \mid e_k : \text{edges, } i_k \in \{\pm 1\} \text{ such that } e_n^{i_n} \cdots e_1^{i_1} \in \pi_1(\Delta; v_0) \right\}.$$

Then under the assumption that the multipliers s_e of all γ_e are sufficiently small,

- Γ is a Schottky group of rank g;

- If $\infty \in \Omega_\Gamma$, then $\sum_{\gamma \in \Gamma} |\gamma'(z)|$ converges uniformly on any compact subset of $\Omega_\Gamma - \bigcup_{\gamma \in \Gamma} \gamma(\infty)$;

- $R_\Gamma = \Omega_\Gamma / \Gamma$ is a Riemann surface of genus g obtained from holed Riemann spheres P_v (v : vertices of Δ) gluing by γ_e (e : edges of Δ);

and hence

$$s_e \to 0 \ (e : \text{edges of } \Delta)$$
$$\Rightarrow \quad R_\Gamma \to \text{the degenerate curve } C_0 = \left(\bigcup_v P_v \right) \Big/ \begin{array}{c} t_e = t_{-e} \\ (e : \text{edges of } \Delta) \end{array} \quad \text{with dual graph } \Delta.$$

Since $\mathbb{P}^1$ has only trivial deformation, R_Γ gives a universal deformation of C_0, and hence varying $t_{\pm e}$ as the **moduli parameters**, s_e as the **deformation parameters**, R_Γ make an open subset (of dimension $3g - 3$ by Exercise 8) of the moduli space of curves of genus g.

Arithmetic Schottky uniformization. An extension of this process in terms of arithmetic geometry (unifying complex geometry and formal geometry over $\mathbf{Z}$, hence rigid geometry) is the following **arithmetic Schottky uniformization theory** which also gives a higher genus version of the Tate curve:

Theorem 4.2. ([I3], (1)–(3) were already proved in [IhN] for maximally degenerate case without singular components). *Let*
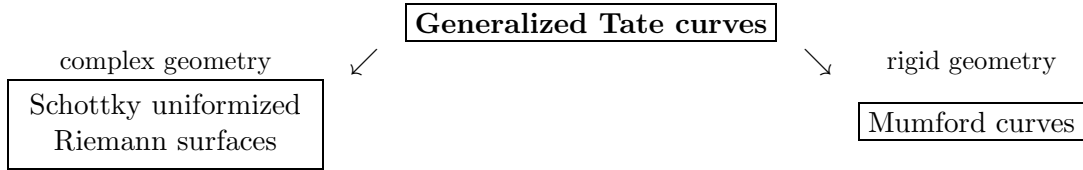
$$A_0 \overset{\text{def}}{=} \text{ the coordinate ring of the moduli space (i.e., the ring of moduli parameters)}$$
$$\text{over } \mathbf{Z} \text{ of degenerate curves with dual graph } \Delta,$$
$$A_\Delta \overset{\text{def}}{=} A_0[[y_e \ (e : \text{edges of } \Delta)]].$$

*Then there exists a stable curve C_Δ (called the **generalized Tate curve**) over A_Δ of genus $g \overset{\text{def}}{=} \text{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z})$ satisfying:*

(1) C_Δ is a universal deformation of the universal degenerate curve with dual graph Δ.

(2) By substituting complex numbers $t_{\pm e}$ to the moduli parameters and $s_e \in \mathbf{C}^\times$ to y_e (e are edges of Δ), C_Δ becomes a Schottky uniformized Riemann surface if s_e are sufficiently small.

(3) C_Δ is smooth over $B_\Delta = A_\Delta\left[1/y_e \ (e : edges \ of \ \Delta)\right]$, and is Mumford uniformized by a Schottky group over B_Δ. Furthermore, for a complete integrally closed noetherian local ring R with quotient field K and a Mumford curve C over $(R \subset)K$ such that Δ is the dual graph of its degenerate reduction, there is a ring homomorphism $A_\Delta \to R$ gives rise to $C_\Delta \otimes_{A_\Delta} K \cong C$.

(4) Using Mumford's theory [Mu3] on degenerating abelian varieties, the generalized Jacobian of C_Δ can be expressed as

$$\mathbf{G}_m^g / \langle (p_{ij})_{1 \le i \le g} \mid 1 \le j \le g \rangle; \quad \mathbf{G}_m \overset{\mathrm{def}}{=} \text{the multiplicative algebraic group,}$$

where the multiplicative periods p_{ij} of C_Δ (called **universal periods**) are given as computable elements of B_Δ.

$$\boxed{\textbf{Generalized Tate curves}}$$

complex geometry $\swarrow$ $\searrow$ rigid geometry

$$\boxed{\begin{array}{c}\text{Schottky uniformized}\\\text{Riemann surfaces}\end{array}} \qquad \boxed{\text{Mumford curves}}$$

<u>Sketch of proof.</u>

- Step 1 of constructing C_Δ is to give a Schottky group Γ_Δ over B_Δ as in the above *general degenerating process*, and show that Γ_Δ is flat over A_Δ (note that this fact together with the result of [Mu2] cannot imply the existence of C_Δ since A_Δ is not local).

- Step 2 is, following argument in [Mu2], to show that the collection of sets consisting of 3 fixed points in $\mathbb{P}^1$ of $\Gamma - \{1\}$ gives rise to a tree which is the universal cover of Δ with covering group Δ, and to construct C_Δ as the quotient by Γ of the glued scheme of $\mathbb{P}^1_{A_\Delta}$ associated with this tree using Grothendieck's formal existence theorem.

- In order to give a power series expansion of p_{ij}, use the infinite product presentation by Schottky [S], Manin and Drinfeld [ManD] of the multiplicative periods given in Theorem 2.2 (3).

Example. When Δ consists of one vertex and g loops, the universal periods p_{ij} are given in 4.1.

Remark. Denote by

$$\begin{array}{rl}
T_g & : \quad \text{the Teichmüller space of degree } g, \\
S_g & : \quad \text{the Schottky space of degree } g \\
& \quad \text{(the moduli space of Schottky groups with free } g \text{ generators),} \\
H_g & : \quad \text{the Siegel upper half space of degree } g.
\end{array}$$

Then

$$
\begin{array}{ccll}
T_g & \xrightarrow{\ p\ } & H_g & \text{: the period map (transcendental)}\\
\downarrow & & \downarrow \exp(2\pi\sqrt{-1}\cdot) & \\
S_g & \longrightarrow & H_g/\mathbf{Z}^{g(g+1)/2} & \text{: computable as power series}\\
\downarrow & & \downarrow & \\
\mathcal{M}_g(\mathbf{C}) & \xrightarrow{\ \tau\ } & H_g/Sp_{2g}(\mathbf{Z}) & \text{: the Torelli map (algebraic).}
\end{array}
$$

Problem. When any vertex of Δ has just 3 branches (i.e., the corresponding degenerate curve is maximally degenerate), the moduli space of degenerate curves with dual graph Δ consists of one point, and hence $A_0 = \mathbf{Z}$. Then express integral coefficients of

$$
p_{ij} \ \in \ A_\Delta = \mathbf{Z}[[y_e \ (e : \text{edges of } \Delta)]]
$$

by using some arithmetic functions (cf. [MaT] for the genus 2 case).

§5. Moduli space of algebraic curves

5.1. Construction of moduli spaces

A **scheme** is a locally ringed space which is locally given by the affine scheme:

$$\mathrm{Spec}(A) \overset{\mathrm{def}}{=} \{\text{prime ideals of } A\} \ni \mathfrak{p} \mapsto A_{\mathfrak{p}} \overset{\mathrm{def}}{=} \{a/s \mid a \in A,\ s \in A - \mathfrak{p}\}$$

associated with a commutative ring A with unit 1 (the category of affine schemes is contravariantly equivalent to that of commutative rings with unit 1). A scheme X over a scheme S is a scheme with morphism $X \to S$.

The **moduli space of curves** is a space representing

the isomorphism classes of curves.

More precisely, if $\mathcal{M}_g$ is a *fine* moduli of curves of genus g, then

$$\mathcal{M}_g(S) \quad \overset{\mathrm{def}}{=} \quad \{\text{morphisms from } S \text{ to } \mathcal{M}_g\} \quad (S : \text{schemes})$$
$$\overset{\text{functorial}}{\cong} \quad \{\text{isomorphism classes of curves over } S \text{ of genus } g\}$$

Caution! There is no fine moduli as an scheme since there are curves with nontrivial automorphism (for example, **hyperelliptic curves** defined by $y^2 = f(x)$ has a nontrivial automorphism $x \mapsto x,\ y \mapsto -y$). Because if $\mathcal{M}_g$ is a fine moduli scheme, then the identity map on $\mathcal{M}_g$ corresponds to a curve $\mathcal{C}$ over $\mathcal{M}_g$ which is universal, i.e., for each scheme S,

$$\{\text{morphisms from } S \text{ to } \mathcal{M}_g\} \quad \longleftrightarrow \quad \{\text{curves over } S \text{ with morphisms to } \mathcal{C}\}$$
$$S \to \mathcal{M}_g \qquad\qquad \longmapsto \qquad\qquad \mathcal{C} \times_{\mathcal{M}_g} S \to \mathcal{C}$$

Therefore, any automorphism on a curve over S must be the identity map.

Solutions.

(S1) Construct the fine moduli as an scheme by considering additional structures on curves.

(S2) Taking the categorical quotient of the above fine moduli, construct the fine moduli as an **algebraic stack**, the scheme-theoretic analog of **orbifolds**, which is represented as

$$[U/R] : \text{ the quotient of } U \text{ by } R,$$

where U, R are schemes with étale morphisms $s, t : R \to U$ and a morphism $\mu : R \times_{U,t,s} R \to R$ such that $(s,t) : R \to U \times U$ is finite and s, t, μ form a groupoid. For a scheme S, $[U/R](S) = \mathrm{Hom}(S, U/R)$ is the category given by

$$\mathrm{Ob}\left([U/R](S)\right) \quad \overset{\mathrm{def}}{=} \quad \mathrm{Hom}(S, U),$$
$$\mathrm{Mor}\left([U/R](S)\right) \quad \overset{\mathrm{def}}{=} \quad \left\{\alpha \in \mathrm{Hom}(S, R) \text{ giving } s \circ \alpha \overset{\sim}{\to} t \circ \alpha\right\},$$

and R gives the equivalence relation by μ.

(S3) Taking the geometric quotient of the above fine moduli, construct the *coarse* moduli as an scheme.

Moduli of elliptic curves.

- **Analytic construction:**

  (S1) If E is an elliptic curve over $\mathbf{C}$, and ι is an isomorphism $\mathbf{Z}^{\oplus 2} \overset{\sim}{\to} H_1(E, \mathbf{Z})$ such that ι is *canonical*, i.e., $\iota(\boldsymbol{e}_1), \iota(\boldsymbol{e}_2)$ intersects as the x, y-axes, then the ratio

  $$\left(\int_{\iota(\boldsymbol{e}_2)} \omega \right) \Big/ \left(\int_{\iota(\boldsymbol{e}_1)} \omega \right)$$

  is independent of $\omega \in H^1(E, \Omega_E) - \{0\}$ and belongs to the Poincaré upper half plane H_1. Therefore, by the correspondence:

  $$H_1 \ni \tau \ \leftrightarrow \ (\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau); \ \iota(\boldsymbol{e}_1) = 1, \iota(\boldsymbol{e}_2) = \tau),$$

  H_1 becomes the fine moduli space of elliptic curves E over $\mathbf{C}$ with canonical isomorphism $\mathbf{Z}^{\oplus 2} \overset{\sim}{\to} H_1(E, \mathbf{Z})$.

  (S2) By (S1), the fine moduli stack of elliptic curves over $\mathbf{C}$ is given by the complex analytic stack, i.e. **orbifold**
  $$[H_1/SL_2(\mathbf{Z})] \, .$$

  (S3) Since

  $$\begin{aligned} & \quad \text{the elliptic curves } y_i^2 = 4x_i^3 - \alpha_i x_i - \beta_i \ (i = 1, 2) \text{ over } \mathbf{C} \text{ are isomorphic} \\ \Leftrightarrow & \quad \text{there are } a, b, c, d, e \in \mathbf{C} \text{ with } a, c \neq 0 \text{ such that} \\ & \quad \begin{cases} x_2 = ax_1 + b : \text{ order} \geq -2 \text{ at the origin,} \\ y_2 = cy_1 + dx_1 + e : \text{ order} \geq -3 \text{ at the origin} \end{cases} \\ \Leftrightarrow & \quad \text{there are } a, c \in \mathbf{C}^\times \text{ such that } a^3 = c^2, x_2 = ax_1, y_2 = cy_1 \\ \Leftrightarrow & \quad \text{the } \boldsymbol{j}\text{-invariants } \frac{\alpha_i^3}{\alpha_i^3 - 27\beta_i^2} \text{ of } y_i^2 = 4x_i^3 - \alpha_i x_i - \beta_i \ (i = 1, 2) \text{ are equal} \\ & \quad \left(\text{note that } \alpha_i^3 - 27\beta_i^2 \neq 0 \right), \end{aligned}$$

  the coarse moduli scheme of elliptic curves over $\mathbf{C}$ becomes the affine line over $\mathbf{C}$, and the j-function

  $$\begin{aligned} j(\tau) \ &\overset{\text{def}}{=} \ \frac{(60E_4(\mathbf{Z} + \mathbf{Z}\tau))^3}{(60E_4(\mathbf{Z} + \mathbf{Z}\tau))^3 - 27(140E_6(\mathbf{Z} + \mathbf{Z}\tau))^2} \\ &= \ \frac{1}{1728} \left(\frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots \right) \left(q \overset{\text{def}}{=} e^{2\pi\sqrt{-1}\tau} \right) \end{aligned}$$

  gives a biholomorphic map from the geometric quotient $H_1/SL_2(\mathbf{Z})$ onto $\mathbf{C}$.

- **Algebraic construction:**

  (S1) For complex numbers $\mu \neq 1$, $\zeta_3 = e^{2\pi\sqrt{-1}/3}$, ζ_3^2, put

  $$E(\mu) \overset{\text{def}}{=} \left\{ (x_0 : x_1 : x_2) \in \mathbb{P}^1(\mathbf{C}) \mid x_0^3 + x_1^3 + x_2^3 = 3\mu x_0 x_1 x_2 \right\} : \textbf{Hesse's cubic}$$
  $$: \text{ an elliptic curve over } \mathbf{C} \text{ with origin } (1 : -1 : 0) \text{ containing}$$
  $$\text{3-division points } (1 : -\beta : 0), (0 : 1 : -\beta), (-\beta : 0 : 1) \ \left(\beta = 1, \zeta_3, \zeta_3^2\right)$$

  Then $\mu \mapsto (E(\mu)$ with the 3-division points) gives a bijection:

  $$\mathbf{C} - \left\{ 1, \zeta_3, \zeta_3^2 \right\} \ \overset{\sim}{\to} \ \left\{ \begin{array}{l} \text{isomorphism classes of elliptic curves over } \mathbf{C} \\ \text{with symplectic level 3 structure} \\ (\mathbf{Z}/3\mathbf{Z})^{\oplus 2} \overset{\sim}{\to} E[3] \overset{\text{def}}{=} \{P \in E \mid 3P = 0\} \end{array} \right\}$$
  $$\cong \ H_1/\Gamma(3),$$

  where $\Gamma(3)$ denotes the principal congruence subgroup of $SL_2(\mathbf{Z})$ of level 3. Therefore, $\mathbf{C} - \left\{ 1, \zeta_3, \zeta_3^2 \right\}$ has a natural model over $\mathbf{Z}\left[1/3, \zeta_3\right]$ as the fine moduli scheme of elliptic curves with level 3 structure, and this can be compactified to $\mathbb{P}^1$ by adding the 4 points $1, \zeta_3, \zeta_3^2, \infty$ which correspond degenerate curves. Nakamura [N] gave this higher dimensional version, i.e., constructed a compactification of the moduli of principally polarized abelian varieties with level structure as an moduli space.

  (S2) The fine moduli stack over $\mathbf{Z}\left[1/3, \zeta_3\right]$ is given by the quotient stack of the above model in (S1) by $SL_2\left(\mathbf{Z}/3\mathbf{Z}\right)$.

- **Scheme theoretic construction:**

  (S1) If E is an elliptic curve over a scheme S with 0-section $e : S \to E$, then

  $$H^0\left(E, \mathcal{O}(3 \cdot e(S))\right) : \text{ the space of generalized elliptic functions}$$

  defines an embedding $E \hookrightarrow \mathbb{P}_S^2$. Therefore, by the theory of Hilbert schemes, there is a fine moduli scheme $\mathcal{H}_1$ over $\mathbf{Z}$ classifying elliptic curves with embedding into $\mathbb{P}_S^2$ as above.

  (S2) The fine moduli stack over $\mathbf{Z}$ of elliptic curves is given by the quotient stack

  $$\left[\mathcal{H}_1/\mathrm{Aut}(\mathbb{P}^2)\right] = \left[\mathcal{H}_1/PGL_3\right]; \quad PGL_n \overset{\text{def}}{=} GL_n/\mathbf{G}_m.$$

Exercise 9. Prove that for $\tau \in H_1$,

$$\{\gamma \in SL_2(\mathbf{Z}) \mid \gamma(\tau) = \tau\}$$

$$= \begin{cases} \left\langle \rho \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \rho^{-1} \right\rangle & : \text{order 6} \quad (\text{if } \exists \rho \in SL_2(\mathbf{Z}) \text{ such that } \tau = \rho(\zeta_3)), \\[2ex] \left\langle \rho \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rho^{-1} \right\rangle & : \text{order 4} \quad (\text{if } \exists \rho \in SL_2(\mathbf{Z}) \text{ such that } \tau = \rho(\sqrt{-1})), \\[2ex] \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle & : \text{order 2} \quad (\text{otherwise}). \end{cases}$$

Construction of moduli for genus > 1. There are 3 approaches using

1. Teichmüller theory: Fix a Riemann surface R_0 of genus $g > 1$. Then the **Teichmüller space** of degree g is defined by

$$T_g \overset{\text{def}}{=} \left\{ (R, h) \;\middle|\; \begin{array}{l} R : \text{Riemann surfaces of genus } g \\ h : \text{orientation preserving diffeomorphisms } R_0 \to R \end{array} \right\} \Big/ \sim$$

$$; \quad (R, h) \sim (R', h') \overset{\text{def}}{\Longleftrightarrow} h' \circ h^{-1} \text{ is homotopic to a biholomorphic map,}$$

and the **Teichmüller modular group** or **mapping class group** of degree g is defined by

$$\Pi_g \overset{\text{def}}{=} \{\text{homotopy classes of orientation preserving diffeomorphisms } R_0 \to R_0\}$$

which acts on T_g as $\mu(R, h) = (R, h \circ \mu)$ ($\mu \in \Pi_g$) properly discontinuously. Then the quotient orbifold $[T_g/\Pi_g]$ exists and becomes the moduli space of Riemann surfaces of genus g. Teichmüller proved that T_g is homeomorphic to $\mathbf{R}^{6g-6}$ and becomes naturally a complex manifold of dimension $3g - 3$ by using the theory of quasiconformal maps (see [IT]). Since T_g is connected and simply connected,

$$\pi_1\left([T_g/\Pi_g]\right) \cong \Pi_g,$$

and these are canonically isomorphic to

$$\mathrm{Aut}^+\left(\pi_1(R_0)\right) \big/ \mathrm{Inn}\left(\pi_1(R_0)\right),$$

where $\mathrm{Aut}^+\left(\pi_1(R_0)\right)$ denotes the automorphism group of $\pi_1(R_0)$ preserving the (alternating and bilinear) intersection form on $H_1\left(R_0, \mathbf{Z}\right) = \pi_1(R_0)/\left[\pi_1(R_0), \pi_1(R_0)\right]$, and $\mathrm{Inn}\left(\pi_1(R_0)\right)$ denotes the inner automorphism group.

Caution! Royden showed that if $g > 1$, then $\mathrm{Aut}(T_g) = \Pi_g$, and hence the T_g is not a homogeneous space. Therefore, one cannot regard the Teichmüller modular group as a discrete subgroup of a Lie group.

2. Moduli theory of abelian varieties: A **principally polarized abelian variety** (A, φ) is a pair of an abelian variety A, i.e., a proper (commutative) algebraic group and an isomorphism $A \to \widehat{A}$ (: the dual abelian variety of A) induced from an ample divisor on A. There exists a moduli space $\mathcal{A}_g$ of principally polarized g-dimensional abelian varieties, and

$$\mathcal{A}_g(\mathbf{C}) \cong \left[H_g / Sp_{2g}(\mathbf{Z})\right].$$

Here

$$H_g \overset{\text{def}}{=} \{Z \in M_g(\mathbf{C}) \mid Z : \text{symmetric, } \mathrm{Im}(Z) > 0\}$$
$$: \quad \text{the } \textbf{Siegel upper half space} \text{ of degree } g,$$
$$Sp_{2g}(\mathbf{Z}) \overset{\text{def}}{=} \left\{G \in M_{2g}(\mathbf{Z}) \;\middle|\; G \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix} {}^t G = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}\right\}$$
$$: \quad \text{the } \textbf{integral symplectic group} \text{ of degree } g \text{ over } \mathbf{Z}$$
$$\text{acts on } H_g \text{ as } Z \mapsto (AZ + B)(CZ + D)^{-1} \text{ for } G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and $Z/\sim\, \in H_g/Sp_{2g}(\mathbf{Z})$ corresponds to the pair of an abelian variety $\mathbf{C}^g/L$, where $L = \mathbf{Z}^g + \mathbf{Z}^g \cdot Z$ is the lattice in $\mathbf{C}^g$ generated by the unit vectors $\boldsymbol{e}_i$ and the i-th row vectors $\boldsymbol{z}_i$ of Z), and the polarization associated with the alternating bilinear form ψ on $L \times L$ such that

$$\psi(\boldsymbol{e}_i, \boldsymbol{e}_j) = \psi(\boldsymbol{z}_i, \boldsymbol{z}_j) = 0, \quad \psi(\boldsymbol{e}_i, \boldsymbol{z}_j) = \delta_{ij}.$$

By Torelli's theorem, by the correspondence:

$$\begin{aligned} &\text{proper smooth curves } C \\ \longmapsto \quad &\text{their Jacobian varieties } \mathrm{Jac}(C) \text{ with principally polarization} \\ &\text{induced from the theta divisor } \{P_1 + \cdots + P_{g-1} - (g-1)P_0 \mid P_i \in C\}, \end{aligned}$$

the (coarse) moduli of proper smooth curves is realized as a subvariety of $\mathcal{A}_g$. This fact gives rise to the **Schottky problem** which means to characterize Jacobian varieties among general abelian varieties, or to describe explicitly the subvariety of $\mathcal{A}_g$ consisting of Jacobian varieties.

3. Geometric invariant theory: For a proper smooth curve C over S of genus $g > 1$, the spaces $H^0\left(C_s, \Omega_{C_s}^{\otimes 3}\right)$ $(s \in S)$ have dimension $5(g-1)$ by Riemann-Roch's theorem, and give an embedding $C \hookrightarrow \mathbb{P}_S^{5g-6}$. Then by the theory of Hilbert schemes, there exists a fine moduli scheme $\mathcal{H}_g$ over $\mathbf{Z}$ classifying tricanonically embedded curves $C \hookrightarrow \mathbb{P}_S^{5g-6}$, and hence the quotient stack

$$\mathcal{M}_g \overset{\mathrm{def}}{=} [\mathcal{H}_g/PGL_{5g-5}]$$

is the fine moduli space of proper smooth curves of genus g. Since PGL_{5g-6} is smooth and the functor $S \mapsto \mathrm{Isom}_S(C, C')$ is represented by a finite and unramified scheme over S for curves C, C' over S, by an étale slice argument, $\mathcal{M}_g$ becomes an algebraic stack. Furthermore, by showing that each point on $\mathcal{H}_g$ is *stable* under the action of PGL_{5g-5}, it follows from **geometric invariant theory** [FKM] by Mumford that the geometric quotient $\mathcal{H}_g/PGL_{5g-5}$ exists and gives the coarse moduli scheme of proper smooth curves of genus g.

Dictionary for the moduli stack. In what follows,

$$\mathcal{M}_g \overset{\mathrm{def}}{=} \text{the moduli stack over } \mathbf{Z} \text{ of proper smooth curves of genus } g > 1.$$

Then

$$\mathcal{M}_g(\mathbf{C}) = \text{the quotient orbifold } [T_g/\Pi_g],$$

and for schemes (more generally algebraic stacks) S,

$$\mathcal{M}_g(S) = \text{the category of proper smooth curves over } S \text{ of genus } g$$
$$\Rightarrow \quad \text{The identity map on } \mathcal{M}_g \text{ gives the } \textbf{universal curve } \mathcal{C} \text{ over } \mathcal{M}_g.$$

Furthermore,

$$\begin{aligned} &\text{an object } \alpha \text{ on (over) } \mathcal{M}_g \\ \Longleftrightarrow \quad &\text{a system } \{\alpha_S\} \text{ of objects on } S \text{ for proper smooth curves over } S \text{ of genus } g \\ &\text{such that } \{\alpha_S\} \text{ are functorial for } S. \end{aligned}$$

Dimension of the moduli.

- **Analytic method:** Since $\mathrm{Aut}(H_1) = PSL_2(\mathbf{R})$, by the theory of Fuchsian models, for Riemann surfaces $R = H_1/\pi_1(R)$, $R' = H_1/\pi_1(R')$ of genus $g > 1$,

$$\big(R; \pi_1(R) \hookrightarrow PSL_2(\mathbf{R})\big) \cong \big(R; \pi_1(R') \hookrightarrow PSL_2(\mathbf{R})\big)$$
$$\Longleftrightarrow \quad \pi_1(R) \text{ and } \pi_1(R') \text{ are conjugate in } PSL_2(\mathbf{R}).$$

Therefore, under fixing a Riemann surface R_0 of genus g,

$$T_g \cong \left\{ \begin{array}{l} \text{conjugacy classes of injective homomorphisms} \\ \iota : \pi_1(R_0) \to PSL_2(\mathbf{R}) \text{ satisfying that} \\ H_1/\iota(\pi_1(R_0)) \text{ are Riemann surfaces of genus } g \end{array} \right\},$$

and the real dimension of the right hand side is

$$\dim_{\mathbf{R}}(PSL_2(\mathbf{R}) \times (\sharp\{\text{generators of } \pi_1(R_0)\} - \sharp\{\text{relations in } \pi_1(R_0)\} - 1)$$
$$= \quad 6g - 6.$$

Furthermore, under the assumption that for Schottky uniformized Riemann surfaces R, R' of genus $g > 1$,

$$\big(R = \Omega_\Gamma/\Gamma; \Gamma \hookrightarrow PGL_2(\mathbf{C})\big) \cong \big(R' = \Omega_{\Gamma'}/\Gamma'; \Gamma' \hookrightarrow PGL_2(\mathbf{C})\big)$$
$$\overset{\text{may be}}{\Longleftrightarrow} \quad \Gamma \text{ and } \Gamma' \text{ are conjugate in } PGL_2(\mathbf{C}),$$

by letting F_g be the free group of rank g, we have

$$\mathcal{M}_g(\mathbf{C}) \cong \left\{ \begin{array}{l} \text{conjugacy classes of injective homomorphisms} \\ \iota : F_g \to PGL_2(\mathbf{C}) \text{ satisfying that} \\ \iota(F_g) \text{ are Schottky groups} \end{array} \right\} \Big/ \mathrm{Aut}(F_g),$$

and the complex dimension of the right hand side is

$$\dim_{\mathbf{C}}(PGL_2(\mathbf{C}) \times (\sharp\{\text{generators of } F_g\} - 1) = 3g - 3.$$

- **Algebraic method (deformation theory [HM]):** For a field k,

$$\begin{array}{rcl} A_0 & \overset{\text{def}}{=} & k[\varepsilon]/(\varepsilon^2), \\ C & : & \text{a proper smooth curve over } k \text{ of genus } g > 1, \\ \{U_\alpha\} & : & \text{an affine open cover of } C, \end{array}$$

and let $\varphi_{\alpha\beta}$ be a first-order infinitesimal deformation of C, i.e., A_0-linear ring homomorphisms

$$\mathcal{O}_{U_\alpha \times \mathrm{Spec}(A_0)}\big|_{(U_\alpha \cap U_\beta)} \to \mathcal{O}_{U_\beta \times \mathrm{Spec}(A_0)}\big|_{(U_\alpha \cap U_\beta)}$$

satisfying that

$$\left\{ \begin{array}{l} \varphi_{\alpha\gamma} = \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \text{ (: the cocycle condition)}, \\ \varphi_{\alpha\beta}\big|_{(U_\alpha \cap U_\beta) \times \mathrm{Spec}(k)} \text{ is the identity.} \end{array} \right.$$

Then the k-linear homomorphisms $D_{\alpha\beta} : \mathcal{O}_{(U_\alpha \cap U_\beta)} \to \mathcal{O}_{(U_\alpha \cap U_\beta)}$ given by $\varphi_{\alpha\beta}(f) = f + \varepsilon D_{\alpha\beta}(f)$ satisfies that

$$D_{\alpha\beta}(f \cdot g) = f \cdot D_{\alpha\beta}(g) + g \cdot D_{\alpha\beta}(f), \quad D_{\alpha\gamma}(f) = D_{\beta\gamma}(f) \cdot D_{\alpha\beta}(f),$$

and hence $\{D_{\alpha\beta}\}$ defines an element of the first cohomology group $H^1(C, \mathcal{T}_C)$ of the tangent bundle $\mathcal{T}_C$ on C. Since $\dim_k(C) = 1$, the obstruction space is $H^2(C, \mathcal{T}_C) = \{0\}$, and hence the tangent space of $\mathcal{M}_g \otimes_{\mathbf{Z}} k$ at the point $[C]$ corresponding to C is isomorphic to $H^1(C, \mathcal{T}_C)$. Therefore,

$$
\begin{aligned}
&\text{the dimension of the tangent space of } \mathcal{M}_g \otimes_{\mathbf{Z}} k \text{ at } [C] \\
=\ & \dim_k H^1\left(C, \mathcal{T}_C\right) \\
=\ & \dim_k H^0\left(C, \Omega_C^{\otimes 2}\right) \quad \text{(by Serre's duality)} \\
=\ & 3g - 3 \quad \text{(by Riemann-Roch's theorem and that } \deg(\Omega_C) = 2g - 2 > 0\text{)}.
\end{aligned}
$$

Remark. For proper smooth curves C,

$$H^1(C, \mathcal{T}_C) \cong \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{T}_C) \cong \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C),$$

and the last group also classifies first-order infinitesimal deformations of stable curves.

5.2. Stable curves and their moduli space

Stable curves. Recall that a **stable curve** of genus $g > 1$ over a scheme S is defined to be a proper and flat morphism $C \to S$ whose geometric fibers are reduced and connected 1-dimensional schemes C_s such that

- C_s has only ordinary double points;

- $\mathrm{Aut}(C_s)$ is a finite group, i.e., if X is a smooth rational component of C_s, then X meets the other components of C_s at least 3 points;

- the dimension of $H^1(C_s, \mathcal{O}_{C_s})$ is equal to g.

For a stable curve C over S (may not be smooth), it is useful to consider the **dualizing sheaf** (or canonical invertible sheaf) $\omega_{C/S}$ on C which is defined as the following conditions:

- $\omega_{C/S}$ is functorial on S;

- if $S = \mathrm{Spec}(k)$ (k is an algebraically closed field), $f : C' \to C$ be the normalization (resolution) of C, $x_1, ..., x_n$, $y_1, ..., y_n$, are the points of C' such that $z_i = f(x_i) = f(y_i)$ ($1 \le i \le n$) are the ordinary double points on C, then $\omega_{C/S}$ is the sheaf of 1-forms η on C' which are regular except for simple poles at x_i, y_i such that

$$\mathrm{Res}_{x_i}(\eta) + \mathrm{Res}_{y_i}(\eta) = 0.$$

Then it is shown by Rosenlicht and Hartshorne that $\omega_{C/S}$ is a line bundle on C, Riemann-Roch's theorem holds for the canonical divisor corresponding to ω_C, and

$$\dim H^1(C_s, \mathcal{O}_{C_s}) = \dim H^0(C_s, \omega_{C_s}).$$

Theorem 5.1. (Deligne and Mumford [DM]) *There exists the fine moduli space $\overline{\mathcal{M}}_g$ (called* **Deligne-Mumford's compactification** *of $\mathcal{M}_g$) as an algebraic stack over $\mathbf{Z}$ classifying stable curves of genus $g > 1$. $\overline{\mathcal{M}}_g$ is proper smooth over $\mathbf{Z}$, and contains $\mathcal{M}_g$ as its open dense substack.*

Sketch of proof. The construction of $\overline{\mathcal{M}}_g$ is similar to that of $\mathcal{M}_g$ by replacing Ω_C with dualizing sheaves ω_C. The properness of $\overline{\mathcal{M}}_g$ follows from the valuative criterion and the **stable reduction theorem**: Let R be a discrete valuation ring with quotient field K, and let C be a proper and smooth curve over K of genus $g > 1$. Then there exists a finite extension L of K and a stable curve $\mathcal{C}$ over the integral closure R_L of R in L such that $\mathcal{C} \otimes_{R_L} L \cong C \otimes_K L$.

Irreducibility of the moduli.

As an application of Theorem 5.1, Deligne and Mumford [DM] proved the irreducibility of any geometric fibers of $\overline{\mathcal{M}}_g$ by applying Enriques-Zariski's connectedness theorem to the proper and smooth stack $\overline{\mathcal{M}}_g$ over $\mathbf{Z}$ whose fiber over $\mathbf{C}$ is connected (by Teichmüller's theory). Therefore,

Any geometric fiber of $\mathcal{M}_g$ is irreducible.

This fact is essentially used in 6.3 to study automorphic forms on the moduli of curves.

5.3. Intersection theory on the moduli space

A cycle class in an algebraic variety X is defined to be a rational equivalence class of $\mathbf{Z}$-linear finite sums of subvarieties of X, and the Chow ring $\mathrm{CH}^*(X)$ denotes the group of cycle classes in X whose ring structure is given by intersection products. The structure of $\mathrm{CH}^*(\mathcal{M}_g)$ is an important subject in algebraic geometry and mathematical physics, and was studied by Mumford, Witten, Kontsevich, Faber, Mirzakhani and others. A basic tool to this study is Grothendieck-Riemann-Roch's theorem for families of algebraic curves.

Grothendieck-Riemann-Roch's theorem (GRR). This theorem states the following: If $\pi : X \to B$ is a proper smooth morphism over a smooth base, and E is a coherent sheaf on X, then

$$\mathrm{ch}\left(\pi_!(E)\right) = \pi_*\left(\mathrm{ch}(E) \cdot \mathrm{td}\left(\mathcal{T}_{C/B}\right)\right)$$

in the Chow ring $\mathrm{CH}^*(B) \otimes_{\mathbf{Z}} \mathbf{Q}$ with $\mathbf{Q}$-coefficients, where ch denotes the exponential Chern character, and $\mathcal{T}_{C/B} = \Omega_{C/B}^{\otimes(-1)}$ denotes the tangent bundle on C over B. In order to apply this theorem to a proper smooth curve $\pi : C \to B$ of genus $g > 1$, and $E = \Omega_{C/B}^n$ ($n \geq 1$), put $\gamma = c_1(\Omega_{C/B})$. Then

$$\mathrm{ch}\left(\pi_*\left(\Omega_{C/B}^n\right)\right) = \pi_*\left(\left(1 + \gamma + \frac{\gamma^2}{2} + \cdots\right)^n \cdot \left(1 - \frac{\gamma}{2} + \frac{\gamma^2}{12} + \cdots\right)\right),$$

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and hence

$$c_0\left(\pi_!\left(\Omega^n_{C/B}\right)\right) = (2n-1)(g-1), \ c_1\left(\pi_*\left(\Omega^n_{C/B}\right)\right) = \frac{6n^2 - 6n + 1}{12}\pi_*\left(\gamma^2\right).$$

The first equality means the original Riemann-Roch's theorem given in 2.2. Furthermore, by putting $n = 1$ in the second equality

$$12c_1\left(\pi_*\left(\Omega^n_{C/B}\right)\right) = (6n^2 - 6n + 1)\cdot\pi_*\left(\gamma^2\right) = 12(6n^2 - 6n + 1)\cdot c_1\left(\pi_*\left(\Omega_{C/B}\right)\right).$$

Since the Picard group of $\mathcal{M}_g$ is torsion-free,

$$c_1\left(\pi_*\left(\Omega^n_{C/B}\right)\right) = (6n^2 - 6n + 1)\cdot c_1\left(\pi_*\left(\Omega_{C/B}\right)\right),$$

and hence we have **Mumford's isomorphism** [Mu4]:

$$\det\left(\pi_*\left(\Omega^n_{C/B}\right)\right) \cong \det\left(\pi_*\left(\Omega_{C/B}\right)\right)^{\otimes(6n^2-6n+1)}$$

between line bundles over B, where $\det(E)$ denotes the determinant line bundle associated with a vector bundle E.

Remark. Morita [Mo] and Mumford [Mu5] conjectured that the stable cohomology groups defined for the moduli spaces of curves over $\mathbf{C}$:

$$H^k(\mathcal{M}) \stackrel{\text{def}}{=} H^k(\mathcal{M}_g(\mathbf{C}), \mathbf{Q}) = H^k(\Pi_g, \mathbf{Q}) \ (g \geq 3k-1)$$
$$: \quad \text{independent of } g \geq 3k-1 \text{ by Harer's result [H2]}$$

satisfies that

$$\bigoplus_{k\geq 0} H^k(\mathcal{M}) \ = \ \mathbf{Q}\left[\kappa_1, \kappa_2, ...\right] : \text{ freely generated over } \mathbf{Q}$$

$$\text{by the \textbf{tautological classes} } \kappa_i = \pi_*\left(\left(c_1\left(\Omega_{\mathcal{C}/\mathcal{M}_g}\right)\right)^{i+1}\right).$$

The free generatedness is proved by Miller [Mi] and Morita [Mo], and the whole conjecture is proved by Madsen and Weiss [MadW].

§6. Arithmetic theory of modular forms

6.1. Elliptic modular forms

The Eisenstein series of even degree $2k \geq 4$ (appeared in the Laurent coefficients of the $\wp$-function $\wp_{\mathbf{Z}+\mathbf{Z}\tau}(z)$):

$$E_{2k}(\tau) \overset{\text{def}}{=} \sum_{(m,n)\in\mathbf{Z}^2-\{(0,0)\}} \frac{1}{(m+n\tau)^{2k}} \overset{\text{Ex.6}}{=} 2\zeta(2k) + \frac{2(2\pi\sqrt{-1})^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) \ q^n$$

$$: \quad \text{the } \textbf{Fourier expansion } \left(q = e^{2\pi\sqrt{-1}\tau}\right)$$

is a holomorphic function of $\tau \in H_1$ which satisfies the following 2 conditions for $SL_2(\mathbf{Z})$:

- Automorphic condition of weight $2k$:
  $$E_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} E_{2k}(\tau) \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z});$$

- Cusp condition :
  $E_{2k}(\tau)$ is holomorphic at $q = 0$ $\left(\Leftrightarrow \tau = \text{ the unique cusp } \sqrt{-1} \cdot \infty \text{ of } SL_2(\mathbf{Z})\right)$.

(Elliptic) modular forms are holomorphic functions on H_1 satisfying the automorphic and cusp conditions for a congruence subgroup of $SL_2(\mathbf{Z})$.

Fourier expansion and number theory. The theory of elliptic modular forms and their Fourier expansions has the following applications to number theory:

1. $\sigma_7(n) = \sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i)$ $\left(\Leftarrow E_8(\tau) = \frac{3}{7}E_4(\tau)^2 \text{ in Exercise 1}\right)$.

2. Jacobi's theorem : $\sharp\left\{(a_i)_{1\leq i\leq 4} \in \mathbf{Z}^4 \ \middle| \ \sum_{i=1}^{4} a_i^2 = n\right\} = 8 \sum_{d|n, 4\nmid d} d$

   $\left(\Leftarrow \text{ the theta series } \left(\sum_{n\in\mathbf{Z}} q^{n^2}\right)^4 \text{ is expressed by Eisenstein series for } \Gamma(2)\right)$.

3. Deligne-Serre's theorem [D, DS]: For a normalized Hecke eigenform $f = \sum_n a(n)q^n$ of weight k and character ε for $\Gamma_0(N)$, there is a 2-dimensional Galois representation ρ_f such that $\text{tr}(\rho_f(F_{\overline{p}})) = a(p)$ and $\det(\rho_f(F_{\overline{p}})) = \varepsilon(p)p^{k-1}$ for any Frobenius automorphism $F_{\overline{p}}$ for unramified primes p.

4. Serre's example [Se]: Let L be the decomposition field of $x^3 - x - 1$ which is a Galois extension over $\mathbf{Q}$ and contains $K = \mathbf{Q}(\sqrt{-23})$ such that the Galois group $\text{Gal}(L/\mathbf{Q})$ is isomorphic to the symmetric group S_3 of degree 3. Put

$$f(\tau) = \frac{1}{2}\left(\sum_{m,n\in\mathbf{Z}} q^{m^2+mn+6n^2} - \sum_{m,n\in\mathbf{Z}} q^{2m^2+mn+3n^2}\right) = \sum_{n=1}^{\infty} a(n)q^n.$$

Then $f(\tau)$ is a normalized Hecke eigenform of weight 1, and hence by Deligne-Serre's theorem, for any prime $p \neq 23$, $\operatorname{tr}(\rho_f(F_{\overline{p}})) = a(p)$, $\det(\rho_f(F_{\overline{p}})) = \left(\dfrac{-23}{p}\right) = \left(\dfrac{p}{23}\right)$ and $\sharp\langle\rho(F_{\overline{p}})\rangle$ is equal to the residue index $f_{L/\mathbf{Q}}(p)$ of p in $L/\mathbf{Q}$. Therefore, this gives an example of nonabelian class field theory.

Exercise 10. Show the above 1.

Exercise 11. Let the notation be as in the above 4. Serre's example. Then prove that for $p \neq 23$, one of the following cases necessarily happens:

$$a(p) = 2, \ \left(\frac{p}{23}\right) = 1 \iff f_{L/\mathbf{Q}}(p) = 1,$$

$$a(p) = 0, \ \left(\frac{p}{23}\right) = -1 \iff f_{K/\mathbf{Q}}(p) = 2, \ f_{L/\mathbf{Q}}(p) = 2,$$

$$a(p) = -1, \ \left(\frac{p}{23}\right) = 1 \iff f_{K/\mathbf{Q}}(p) = 1, \ f_{L/\mathbf{Q}}(p) = 3,$$

and describe the decomposition of primes $2, 3, 5, 59$ in K and L respectively.

Rationality of modular forms. For $\tau \in H_1$,

$$E_\tau \overset{\mathrm{def}}{=} \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) \text{ define a family of elliptic curves over } H_1,$$

$$z_\tau \overset{\mathrm{def}}{=} \text{ the natural coordinate of } \mathbf{C}$$

$$\Rightarrow \ dz_\tau : \text{a canonical base of } H^0(E_\tau, \Omega_{E_\tau}),$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$$

$$\Rightarrow \ E_{\frac{a\tau+b}{c\tau+d}} \overset{\times(c\tau+d)}{\longrightarrow} \mathbf{C}/(\mathbf{Z}(c\tau+d) + \mathbf{Z}(a\tau+b)) = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) = E_\tau$$

$$\Rightarrow \ dz_{\frac{a\tau+b}{c\tau+d}} = \frac{1}{c\tau+d}dz_\tau.$$

If $f(\tau)$ is a modular form of weight k for $SL_2(\mathbf{Z})$, then

$$f\left(\frac{a\tau+b}{c\tau+d}\right)\left(d_{\frac{a\tau+b}{c\tau+d}}\right)^{\otimes k} = (c\tau+d)^k f(\tau)\left(\frac{1}{c\tau+d}\right)^k (dz_\tau)^{\otimes k} = f(\tau)(dz_\tau)^{\otimes k},$$

and hence $f(\tau)dz_\tau$ ($\tau \in H_1$) is $SL_2(\mathbf{Z})$-invariant, i.e., defines a holomorphic section of the line bundle on $[H_1/SL_2(\mathbf{Z})]$ whose fiber over $\tau \in H_1$ is given by $H^0(E_\tau, \Omega_{E_\tau})^{\otimes k}$.

Let $\mathcal{M}_1$ be the moduli stack of elliptic curves, $\pi : \mathcal{E} \to \mathcal{M}_1$ be the universal elliptic curve, and $\pi_*(\Omega_{\mathcal{E}/\mathcal{M}_1})$ denote a line bundle on $\mathcal{M}_1$ defined by the direct image of the sheaf $\Omega_{\mathcal{E}/\mathcal{M}_1}$ of relative 1-forms on $\mathcal{E}/\mathcal{M}_1$, i.e.,

$$\pi_*(\Omega_{\mathcal{E}/\mathcal{M}_1})(S) \overset{\mathrm{def}}{=} H^0(E, \Omega_{E/S}),$$

for elliptic curves E over schemes S. Then an **integral modular form** f of weight k is defined as an element of

$$H^0\left(\mathcal{M}_1, \pi_*\left(\Omega_{\mathcal{E}/\mathcal{M}_1}\right)^{\otimes k}\right),$$

i.e., a global section of $\pi_*(\Omega_{\mathcal{E}/\mathcal{M}_1})^{\otimes k}$ on $\mathcal{M}_1$ which is, by the above dictionary on the moduli stack, a system of

$$\left\{\text{sections } f_S \text{ of } H^0\left(E, \Omega_{E/S}\right)^{\otimes k} \mid E : \text{elliptic curves over } S\right\}$$

which are functorial for schemes S. Hence

$$E/S \ : \ \text{the Tate curve } y^2 + xy = x^3 + a_4(q)x + a_6(q) \text{ over } \mathbf{Z}((q))$$

$$\Rightarrow \quad \frac{du}{u} = \frac{dX(u,q)}{X(u,q) + 2Y(u,q)} = \frac{dx}{x + 2y} \ : \ \text{a base of 1-forms on the Tate curve}$$

$$\Rightarrow \quad f \text{ is represented as } F(f)\left(\frac{dx}{x+2y}\right)^{\otimes k},$$

where $F(f) \in \mathbf{Z}((q))$ is called the **evaluation** of f on the Tate curve under the trivialization of $\pi_*\left(\Omega_{\mathcal{E}/\mathcal{M}_1}\right)$ on $\mathbf{Z}((q))$. By Theorem 4.1 (2),

$$q = e^{2\pi\sqrt{-1}\tau} \quad \Rightarrow \quad \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) \cong \mathbf{C}^\times/\langle q\rangle$$

$$\Rightarrow \quad \frac{dx}{x+2y} = 2\pi\sqrt{-1}\frac{d\wp_{\mathbf{Z}+\mathbf{Z}\tau}(z_\tau)}{\wp'_{\mathbf{Z}+\mathbf{Z}\tau}(z_\tau)} = 2\pi\sqrt{-1}dz_\tau$$

$$\Rightarrow \quad f(\tau) = (2\pi\sqrt{-1})^k F(f)(dz_\tau)^{\otimes k}.$$

Therefore, ignoring the factor $(2\pi\sqrt{-1})^k$,

the evaluation on the Tate curve = the classical Fourier expansion,

and hence

a modular form is integral $\Longleftrightarrow$ its Fourier coefficients are integral.

Exercise 12.

- Prove that $\dfrac{E_4(\tau)}{2\zeta(4)}$, $\dfrac{E_6(\tau)}{2\zeta(6)}$ and $\Delta(\tau) \overset{\text{def}}{=} \dfrac{1}{1728}\left(\left(\dfrac{E_4(\tau)}{2\zeta(4)}\right)^3 - \left(\dfrac{E_6(\tau)}{2\zeta(6)}\right)^2\right)$ are integral (elliptic) modular forms for $SL_2(\mathbf{Z})$.

- Using that $\Delta(\tau) \neq 0$ ($\tau \in H$) and that modular forms for $SL_2(\mathbf{Z})$ of weight 0 are constant, prove that all integral modular forms for $SL_2(\mathbf{Z})$ are generated over $\mathbf{Z}$ by these 3 modular forms.

6.2. Siegel modular forms (SMFs)

Moduli of abelian varieties. Let g be a positive integer > 1. Then in a similar way to constructing moduli of curves given in 5.1, it is shown in [FKM] that there exists the fine

moduli space $\mathcal{A}_g$ as an algebraic stack over $\mathbf{Z}$ classifying principally polarized abelian varieties of dimension g. By the correspondence:

$$Z = \begin{pmatrix} \boldsymbol{z}_1 \\ \vdots \\ \boldsymbol{z}_g \end{pmatrix} \in H_g : \text{ the Siegel upper half space of degree } g$$

$$\leftrightarrow \ \left(\mathbf{C}^g/(\mathbf{Z}^g + \mathbf{Z}^g \cdot Z); \iota(\boldsymbol{e}_i) = \left\{ \begin{array}{ll} \boldsymbol{e}_i & (1 \leq i \leq g), \\ \boldsymbol{z}_{i-g} & (g+1 \leq i \leq 2g) \end{array} \right. \right),$$

H_g becomes the fine moduli space of principally polarized abelian varieties X of dimension g over $\mathbf{C}$ with symplectic isomorphism $\mathbf{Z}^{2g} \overset{\sim}{\to} H_1(X, \mathbf{Z})$. Hence the orbifold $\mathcal{A}_g(\mathbf{C})$ is given by the quotient stack of H_g by the integral symplectic group $Sp_{2g}(\mathbf{Z})$ of degree g :

$$\mathcal{A}_g(\mathbf{C}) \ = \ [H_g/Sp_{2g}(\mathbf{Z})] \, .$$

<u>Definition of SMFs.</u> Let λ be the **Hodge line bundle** on $\mathcal{A}_g$ which is defined by

$$\lambda \overset{\text{def}}{=} \bigwedge^g \rho_* \left(\Omega_{\mathcal{X}/\mathcal{A}_g} \right) \quad (\rho : \mathcal{X} \to \mathcal{A}_g \text{ denotes the universal abelian scheme})$$

$$\Rightarrow \ \lambda(S) = \bigwedge^g H^0 \left(X, \Omega_{X/S} \right) \quad \text{for abelian schemes } X/S \text{ of relative dimension } g.$$

Then for $h \in \mathbf{Z}$ and a $\mathbf{Z}$-module M, we call elements of

$$S_{g,h}(M) \overset{\text{def}}{=} H^0 \left(\mathcal{A}_g, \lambda^{\otimes h} \otimes_{\mathbf{Z}} M \right)$$

Siegel modular forms of degree g and weight h with coefficients in M.

For the natural coordinate $z_1, ..., z_g$ on the complex abelian varieties

$$X_Z = \mathbf{C}^g/(\mathbf{Z}^g + \mathbf{Z}^g \cdot Z),$$

$dz_1, ..., dz_g$ give a base of $H^0(X_Z, \Omega_{X_Z})$, and hence as in the elliptic case,

$$\varphi = (2\pi\sqrt{-1})^{gh} \cdot f \cdot (dz_1 \wedge \cdots \wedge dz_g)^{\otimes h} \in S_{g,h}(\mathbf{C}) = H^0 \left([H_g/Sp_{2g}(\mathbf{Z})], \lambda^{\otimes h} \right)$$

$$\overset{(*)}{\Rightarrow} \left\{ \begin{array}{l} f = f(Z) \text{ is a holomorphic function of } Z \in H_g \text{ such that} \\ f(G(Z)) = \det(CZ + D)^h \cdot f(Z) \text{ for any } G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbf{Z}) \end{array} \right.$$

which is known as the usual definition of **analytic Siegel modular forms**. In particular, $f(Z)$ is invariant under the transformation

$$Z \longmapsto Z + B$$

by integral symmetric matrices B of degree g, and hence it can be expanded as a power series of $\exp\left(2\pi\sqrt{-1}z_{ij}\right)$ $(Z = (z_{ij})_{i,j} \in H_g)$ which is called the **(classical) Fourier expansion** of f.

Exercise 13. Prove the above $\overset{(*)}{\Rightarrow}$.

It is shown by Satake that $\mathcal{A}_{g/\mathbf{C}} = \mathcal{A}_g \otimes_{\mathbf{Z}} \mathbf{C}$ has the **Satake compactification**:

$$\mathcal{A}^*_{g/\mathbf{C}} = \coprod_{i=0}^{g} \mathcal{A}_{i/\mathbf{C}},$$

obtained as the Zariski closure of a projective embedding using Siegel modular forms of sufficiently large weight. Then the codimension of $\mathcal{A}^*_{g/\mathbf{C}} - \mathcal{A}_{g/\mathbf{C}}$ in $\mathcal{A}^*_{g/\mathbf{C}}$ is

$$\frac{g(g+1)}{2} - \frac{(g-1)g}{2} = g > 1,$$

and hence ignoring $(2\pi\sqrt{-1})^{gh} (dz_1 \wedge \cdots \wedge dz_g)^{\otimes h}$,

φ is an analytic Siegel modular form

$\Rightarrow$ φ is an analytic section on $\mathcal{A}^*_{g/\mathbf{C}}$ (by Hartogs' theorem)

$\Rightarrow$ φ is an algebraic section on $\mathcal{A}^*_{g/\mathbf{C}}$ (by GAGA's principle of Serre)

$\Rightarrow$ φ is an algebraic section on $\mathcal{A}_{g/\mathbf{C}}$

$\Rightarrow$ $\varphi \in S_{g,h}(\mathbf{C})$.

Therefore, the above $\overset{(*)}{\Rightarrow}$ is in fact an equivalence $\overset{(*)}{\Longleftrightarrow}$, and $S_{g,h}(\mathbf{C})$ is finite dimensional over $\mathbf{C}$ by the compactness of $\mathcal{A}^*_{g/\mathbf{C}}$.

Fourier expansion of SMFs. By Mumford's theory [Mu3] on degenerating abelian varieties, there exists a semiabelian scheme expressed as

$$\mathbf{G}_m^g / \langle (q_{ij})_{1 \leq i \leq g} \mid 1 \leq j \leq g \rangle$$

over the ring

$$\mathbf{Z}\left[q_{ij}^{\pm 1} \ (i \neq j) \right] [[q_{11}, ..., q_{gg}]],$$

where q_{ij} $(1 \leq i, j \leq g)$ are variables with symmetry $q_{ij} = q_{ji}$. This semiabelian scheme gives a family of complex abelian varieties

$$\mathbf{C}^g / (\mathbf{Z} + \mathbf{Z} \cdot Z) \cong (\mathbf{C}^\times)^g \Big/ \Big\langle \big(\exp(2\pi\sqrt{-1}z_{ij}) \big)_{1 \leq i \leq g} \mid 1 \leq j \leq g \Big\rangle$$

when $q_{ij} = \exp(2\pi\sqrt{-1}z_{ij})$ for $Z = (z_{ij})_{i,j} \in H_g$. Then the natural coordinates $u_1, ..., u_g$ on $\mathbf{G}_m^g$ give a base $du_1/u_1, ..., du_g/u_g$ of 1-forms on this semiabelian scheme, and hence the evaluation of any $\varphi \in S_{g,h}(M)$ gives

$$\begin{aligned}
\varphi &= F(\varphi) \cdot ((du_1/u_1) \wedge \cdots \wedge (du_g/u_g))^{\otimes h} \\
&= (2\pi\sqrt{-1})^{gh} \cdot F(\varphi) \cdot (dz_1 \wedge \cdots \wedge dz_g)^{\otimes h} \quad \text{(if } M = \mathbf{C} \text{ and } u_i = \exp(2\pi\sqrt{-1}z_i)).
\end{aligned}$$

Therefore, we have a linear map:

$$F : S_{g,h}(M) \longrightarrow \mathbf{Z}\left[q_{ij}^{\pm 1} \ (i \neq j) \right] [[q_{11}, ..., q_{gg}]] \otimes_{\mathbf{Z}} M,$$

which we call the **arithmetic Fourier expansion.**

Theorem 6.1. (Chai and Faltings [FaC])

(1) *(Arithmetic Fourier expansion) F is functorial for M, and if $M = \mathbf{C}$, then $F(\varphi)$ is the classical Fourier expansion by $q_{ij} = \exp\left(2\pi\sqrt{-1}z_{ij}\right)$ for $(z_{ij})_{i,j} \in H_g$. Furthermore, F is injective, and for a submodule N of M and $\varphi \in S_{g,h}(M)$,*

$$\varphi \in S_{g,h}(N) \iff F(\varphi) \in \mathbf{Z}\left[q_{ij}^{\pm 1}\right][[q_{ii}]] \otimes_{\mathbf{Z}} M.$$

(2) *(Finiteness) $S_{g,h}(\mathbf{Z})$ is a free $\mathbf{Z}$-module of finite rank such that $S_{g,h}(\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = S_{g,h}(\mathbf{C})$ and that $S_{g,0}(\mathbf{Z}) = \mathbf{Z}$, $S_{g,h}(\mathbf{Z}) = \{0\}$ if $n < 0$. Furthermore, the ring of integral Siegel modular forms of degree g over $\mathbf{Z}$:*

$$S_g^*(\mathbf{Z}) \stackrel{\text{def}}{=} \bigoplus_{h \geq 0} S_{g,h}(\mathbf{Z})$$

is a normal ring finitely generated over $\mathbf{Z}$.

Sketch of Proof. (1) The functoriality for M and the compatibility with the classical Fourier expansion is clear from the above. Since $\mathcal{A}_g$ is smooth over $\mathbf{Z}$, we have the following left exactness of $S_{g,h}(M)$ for M :

$$0 \to N \to M \to (M/N) \to 0$$
$$\Rightarrow \quad 0 \to \lambda^{\otimes h} \otimes_{\mathbf{Z}} N \to \lambda^{\otimes h} \otimes_{\mathbf{Z}} M \to \lambda^{\otimes h} \otimes_{\mathbf{Z}} (M/N) \to 0$$
$$\Rightarrow \quad 0 \to S_{g,h}(N) \to S_{g,h}(M) \to S_{g,h}(M/N).$$

We prove the injectivity of F. Since any $\mathbf{Z}$-module M is the direct limit of finitely generated $\mathbf{Z}$-modules, and cohomology and tensor product commute with direct limit, we may assume that M is a finitely generated $\mathbf{Z}$-module, hence by the left exactness for M, we may put $M = \mathbf{Z}$ or $= \mathbf{Z}/p\mathbf{Z}$ (p : a prime number). Therefore, the injectivity follows from that $\mathcal{A}_g \otimes M$ is smooth over the ring M with geometrically irreducible fibers which is proved in [FaC]. Hence the remains of (1) follows from this injectivity and the left exactness of $S_{g,h}$.

(2) is derived by the following result in [FaC]: there exists an algebraic stack $\overline{\mathcal{A}}_g$ which is proper smooth over $\mathbf{Z}$ and contains $\mathcal{A}_g$ as its open dense substack, and any integral Siegel modular form of weight k can be extended to a section on $\overline{\mathcal{A}}_g$ of an extension $\overline{\lambda}^{\otimes k}$ of $\lambda^{\otimes k}$ (called **Koecher's principle**).

The finiteness of $\text{rank}_{\mathbf{Z}} S_{g,h}(\mathbf{Z})$ follows from these results immediately. Further, there is $m \in \mathbf{N}$ such that $\overline{\lambda}^{\otimes m}$ defines a projective morphism $\overline{\mathcal{A}}_g \to \mathbb{P}_{\mathbf{Z}}^n$ which can be, by the theory of Stein factorization, decomposed as $\overline{\mathcal{A}}_g \to \mathcal{A}_g^* \to \mathbb{P}_{\mathbf{Z}}^n$ such that $\overline{\mathcal{A}}_g \to \mathcal{A}_g^*$ has connected geometric fibers and $\mathcal{A}_g^* \to \mathbb{P}_{\mathbf{Z}}^n$ is finite. Therefore, replacing m by a multiple $\overline{\lambda}^{\otimes m}$ defines a immersion of $\mathcal{A}_g^*$, and hence $\bigoplus_{k \geq 0} H^0(\overline{\mathcal{A}}_g, \overline{\lambda}^{\otimes mk})$ and $S_g^*(\mathbf{Z})$ are normal rings finitely generated over $\mathbf{Z}$. QED.

Ring of SMFs of degree 2 and 3. (Igusa [Ig1,3], Tsuyumine [Ty1]) For $g > 1$ and $h > g + 1$, the **Eisenstein series** of degree $g > 1$ and weight h is a function of $Z \in H_g$ defined by

$$E_{g,h}(Z) \stackrel{\text{def}}{=} \sum_{G \in \Gamma_\infty \backslash Sp_{2g}(\mathbf{Z})} \det(CZ + D)^{-h}; \quad G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where
$$\Gamma_\infty \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} U & B \\ 0 & {}^t U^{-1} \end{pmatrix} \in Sp_{2g}(\mathbf{Z}) \right\}.$$

Then $E_{g,h}$ becomes a Siegel modular form with Fourier coefficients in $\mathbf{Q}$, and hence an element of $S_{g,h}(\mathbf{Q})$. Igusa [Ig1] proved that

$$S_2^*(\mathbf{C}) \ = \ \mathbf{C}\left[E_4, E_6, \Delta_{10}, \Delta_{12}\right] \bigoplus \Delta_{35} \cdot \mathbf{C}\left[E_4, E_6, \Delta_{10}, \Delta_{12}\right],$$

where $E_h = E_{2,h}$, $\Delta_{10} = E_4 E_6 - E_{10}$, $\Delta_{12} = 441 E_4^3 + 250 E_6^2 - 691 E_{12}$ and $\Delta_{35} \in S_{2,35}(\mathbf{C})$ is given by Ibukiyama as

$$\Delta_{35}\begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} = \begin{vmatrix} 4E_4 & 6E_6 & 10\Delta_{10} & 12\Delta_{12} \\[2mm] \dfrac{\partial E_4}{\partial z_{11}} & \dfrac{\partial E_6}{\partial z_{11}} & \dfrac{\partial \Delta_{10}}{\partial z_{11}} & \dfrac{\partial \Delta_{12}}{\partial z_{11}} \\[2mm] \dfrac{\partial E_4}{\partial z_{12}} & \dfrac{\partial E_6}{\partial z_{12}} & \dfrac{\partial \Delta_{10}}{\partial z_{12}} & \dfrac{\partial \Delta_{12}}{\partial z_{12}} \\[2mm] \dfrac{\partial E_4}{\partial z_{22}} & \dfrac{\partial E_6}{\partial z_{22}} & \dfrac{\partial \Delta_{10}}{\partial z_{22}} & \dfrac{\partial \Delta_{12}}{\partial z_{22}} \end{vmatrix}.$$

In [I5], this result was extended to $S_2^*(R)$, where R is a $\mathbf{Z}$-algebra in which 6 is invertible.

Igusa [Ig3] determined generators of $S_2^*(\mathbf{Z})$, and Tsuyumine [Ty1] gave explicit generators of $S_3^*(\mathbf{C})$.

6.3. Teichmüller modular forms (TMFs)

Analytic	:	automorphic functions on the Teichmüller space
	=	automorphic forms on the moduli space of Riemann surfaces,
Algebraic	:	global sections of line bundles on the moduli of curves.

This naming is an analogy of

> Siegel modular forms (SMFs)
> = automorphic functions on the Siegel upper half space
> = global sections of line bundles
> on the moduli of principally polarized abelian varieties.

<u>Definition of TMFs.</u> Let $\pi : \mathcal{C} \to \mathcal{M}_g$ be the universal curve over the moduli stack of proper smooth curves of genus $g > 1$, and let $\lambda \stackrel{\text{def}}{=} \bigwedge^g \pi_* \left(\Omega_{\mathcal{C}/\mathcal{M}_g}\right)$ be the **Hodge line bundle**. Then for a $\mathbf{Z}$-module M, we call elements of

$$T_{g,h}(M) \stackrel{\text{def}}{=} H^0(\mathcal{M}_g, \lambda^{\otimes h} \otimes_{\mathbf{Z}} M)$$

Teichmüller modular forms of degree g and weight h with coefficients in M. By the pullback of the Torelli map $\tau : \mathcal{M}_g \to \mathcal{A}_g$ sending curves to their Jacobian varieties with canonical polarization, we have a linear map

$$\tau^* : S_{g,h}(M) \longrightarrow T_{g,h}(M)$$

for **Z**-modules M. If $g = 2, 3$, then the image of the Torelli map is Zariski dense, and hence τ^* is injective.

If $n \geq 3$, then

$$\mathcal{M}_{g,n/\mathbf{C}} \overset{\text{def}}{=} \quad \text{the moduli space of proper smooth curves over } \mathbf{C}$$
$$\text{of genus } g \text{ with symplectic level } n \text{ structure,}$$

$$\mathcal{A}_{g,n/\mathbf{C}} \overset{\text{def}}{=} \quad \text{the moduli space of principally polarized abelian varieties over } \mathbf{C}$$
$$\text{of dimension } g \text{ with symplectic level } n \text{ structure}$$

are given as fine moduli schemes over **C**. Let $\mathcal{M}^*_{g,n/\mathbf{C}}$ be the **Satake-type** compactification, i.e., normalization of the Zariski closure of

$$(\iota \circ \tau)(\mathcal{M}_{g,n/\mathbf{C}}) \subset \mathcal{A}^*_{g,n/\mathbf{C}},$$

where $\tau : \mathcal{M}_{g,n/\mathbf{C}} \to \mathcal{A}_{g,n/\mathbf{C}}$ denote the Torelli map, and $\iota : \mathcal{A}_{g,n/\mathbf{C}} \to \mathcal{A}^*_{g,n/\mathbf{C}}$ denote the natural inclusion to the Satake compactification. Then each point of $\mathcal{M}^*_{g,n/\mathbf{C}} - \mathcal{M}_{g,n/\mathbf{C}}$ corresponds to the product $J_1 \times \cdots \times J_m$ of Jacobian varieties over **C** with canonical polarization and symplectic level n structure such that $\sum_{i=1}^m \dim(J_i) \leq g$ and that $(m, g) \neq (1, \dim(J_1))$. Therefore, if $g \geq 3$, then $\mathcal{M}^*_{g,n/\mathbf{C}} - \mathcal{M}_{g,n/\mathbf{C}}$ has codimension 2 in $\mathcal{M}^*_{g,n/\mathbf{C}}$, and hence by applying Hartogs' theorem to $\mathcal{M}_{g,n/\mathbf{C}} \subset \mathcal{M}^*_{g,n/\mathbf{C}}$ and GAGA's principle to $\mathcal{M}^*_{g,n/\mathbf{C}}$, one can see that analytic TMFs become algebraic TMFs, i.e.,

$$T_{g,h}(\mathbf{C}) \cong \left\{ \begin{array}{l} \text{holomorphic functions on the Teichmüller space } T_g \\ \text{with automorphic condition of weight } h \\ \text{for the action of the Teichmüller modular group } \Pi_g \end{array} \right\},$$

and this space is finite dimensional over **C**.

Exercise 14. Give a precise definition of analytic Teichmüller modular forms.

Expansion of TMFs. Let C_Δ be the generalized Tate curve given in Theorem 4.2 which is smooth over the ring B_Δ. Then as in the elliptic and Siegel modular case, the evaluation on C_Δ (= the expansion by the corresponding local coordinates on $\mathcal{M}_g$) gives rise to a homomorphism

$$\kappa_\Delta : T_{g,h}(M) \longrightarrow B_\Delta \otimes_{\mathbf{Z}} M.$$

Theorem 6.2. ([I3]). *Fix $g > 1$ and $h \in \mathbf{Z}$.*

 (1) κ_Δ is injective, and for a Teichmüller modular form $f \in T_{g,h}(M)$ and a submodule N of M,

$$f \in T_{g,h}(N) \iff \kappa_\Delta(f) \in B_\Delta \otimes_{\mathbf{Z}} N.$$

 (2) For a Siegel modular form $\varphi \in S_{g,h}(M)$,

$$\kappa_\Delta(\tau^*(\varphi)) = F(\varphi)|_{q_{ij}=p_{ij}},$$

where p_{ij} are the multiplicative periods of C_Δ given in Theorem 4.2 (4).

 Proof. (1) follows from the fact that C_Δ corresponds to the generic point on $\mathcal{M}_g$, and the argument in the proof of Theorem 6.1 (1) replacing $\mathcal{A}_g$ by $\mathcal{M}_g$ which is proper and smooth

over $\mathbf{Z}$ with geometrically irreducible fibers (see 5.2). (2) follows from Theorem 4.2 (4). QED.

Schottky problem. As an application of Theorem 6.2, we can give a solution to the Schottky problem, i.e. characterizing Siegel modular forms vanishing on the Jacobian locus, is given as follows:

$$\tau^*(\varphi) = 0 \iff F(\varphi)|_{q_{ij}=p_{ij}} = 0.$$

$$\boxed{p_{ij} \text{ are computable, hence } \kappa_\Delta \text{ are computable}}$$

Using the universal periods p_{ij} given in Example 4.1, the above implies the following result of Brinkmann and Gerritzen [BG, G]: For the Fourier expansion

$$F(\varphi) = \sum_{T=(t_{ij})} a_T \prod_{1 \leq i < j \leq g} q_{ij}{}^{2t_{ij}} \prod_{1 \leq i \leq g} q_{ii}{}^{t_{ii}}$$

of a Siegel modular form φ vanishing on the Jacobian locus,

$$\text{integers } s_1, ..., s_g \geq 0 \text{ satisfy } \sum_{i=1}^{g} s_i = \min\{\mathrm{T}(T) \mid a_T \neq 0\}$$

$$\Rightarrow \quad \sum_{t_{ii}=s_i} a_T \prod_{i<j} \left(\frac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_i - x_{-j})(x_{-i} - x_j)} \right)^{2t_{ij}} = 0 \text{ in } A_0 \text{ (: given in Example 4.1).}$$

Schottky's J. For $n \equiv 0 \mod(4)$, put

$$L_{2n} \overset{\text{def}}{=} \left\{ (x_1, ..., x_{2n}) \in \mathbf{R}^{2n} \,\middle|\, 2x_i, \ x_i - x_j, \ \frac{1}{2}\sum_i x_i \in \mathbf{Z} \right\}$$

$: \quad$ a lattice in $\mathbf{R}^{2n}$ with standard inner product,

$$\varphi_n(Z) \overset{\text{def}}{=} \sum_{(\lambda_1,...,\lambda_4) \in L_{2n}^4} \exp\left(\pi\sqrt{-1} \sum_{i,j=1}^{4} \langle \lambda_i, \lambda_j \rangle z_{ij} \right) \quad (Z = (z_{ij})_{i,j} \in H_4)$$

$: \quad$ a Siegel modular form of degree 4 and weight n,

$$J(Z) \overset{\text{def}}{=} \frac{2^2}{3^2 \cdot 5 \cdot 7} (\varphi_4(Z)^2 - \varphi_8(Z)) : \textbf{ Schottky's } \boldsymbol{J}$$

$: \quad$ an integral Siegel modular form of degree 4 and weight 8.

Then Schottky and Igusa proved that the Zariski closure of the Jacobian locus in $\mathcal{A}_4 \otimes_{\mathbf{Z}} \mathbf{C}$ is defined by $J = 0$.

Brinkmann and Gerritzen [BG, G] checked the above Brinkmann and Gerritzen's criterion for Schottky's J, i.e., computed the lowest term of J and showed that this is given by up to a constant

$$F \frac{q_{11}q_{22}q_{33}q_{44}}{\prod_{1 \leq i < j \leq 4} q_{ij}},$$

where F is a generator of the ideal of $\mathbf{C}\left[q_{ij}\ (1 \leq i < j \leq 4)\right]$ which is the kernel of the ring homomorphism given by

$$q_{ij} \mapsto \frac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_i - x_{-j})(x_{-i} - x_j)} \in A_0.$$

Problem. Let J' be a primitive modular form obtained from J by dividing the GCM (greatest common divisor) of its Fourier coefficients. Then for each prime p,

the closed subset of $\mathcal{A}_4 \otimes_{\mathbf{Z}} \mathbf{F}_p$ defined by $J' \mod(p) = 0$

$\overset{?}{\doteq}$ the Zariski closure of $\tau(\mathcal{M}_4 \otimes_{\mathbf{Z}} \mathbf{F}_p)$ in $\mathcal{A}_4 \otimes_{\mathbf{Z}} \mathbf{F}_p$.

Hyperelliptic Schottky problem. ([I4]) Let p_{ij} be the universal periods given in Example 4.1. Then

$$p'_{ij} \overset{\text{def}}{=} p_{ij}|_{x_{-k}=-x_k} \quad (1 \leq k \leq g)$$

become the multiplicative periods of the hyperelliptic curve C_{hyp} over

$$\mathbf{Z}\left[\frac{1}{2x_i}, \ \frac{1}{x_i \pm x_j}(i \neq j)\right][[y_1, ..., y_g]]$$

uniformized by the Schottky group:

$$\left\langle \left(\begin{array}{cc} x_k & -x_k \\ 1 & 1 \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & y_k \end{array}\right)\left(\begin{array}{cc} x_k & -x_k \\ 1 & 1 \end{array}\right)^{-1} \ \middle| \ k = 1, ..., g \right\rangle.$$

Since C_{hyp} is generic in the moduli space of hyperelliptic curves, for any Siegel modular form φ over a field of characteristic $\neq 2$,

φ vanishes on the locus of hyperelliptic Jacobians $\iff F(\varphi)|_{q_{ij}=p'_{ij}} = 0.$

Problem. Give an explicit lower bound of $n(g) \in \mathbf{N}$ satisfying that

φ vanishes on the locus of hyperelliptic Jacobians $\iff F(\varphi)|_{q_{ij}=p'_{ij}} \in I^{n(g)},$

where I is the ideal generated by $y_1, ..., y_g$.

Theta constants and ring structure.

For $g \geq 2$, let

$$\theta_g(Z) \overset{\text{def}}{=} \prod_{\substack{\boldsymbol{a}, \boldsymbol{b} \,\in\, \{0, 1/2\}^g \\ 4\boldsymbol{a}^t\boldsymbol{b} \,:\, \text{even}}} \sum_{\boldsymbol{n}\in\mathbf{Z}^g} \exp\left(2\pi\sqrt{-1}\left[\frac{1}{2}(\boldsymbol{n} + \boldsymbol{a})Z^t(\boldsymbol{n} + \boldsymbol{a}) + (\boldsymbol{n} + \boldsymbol{a})^t\boldsymbol{b}\right]\right)$$

be the product of even **theta constants** of degree g. If $g \geq 3$, then θ_g is an integral Siegel modular form of degree g and weight $2^{g-2}(2^g + 1)$.

Theorem 6.3. ([I2, 3]). *For $g \geq 3$,*

(1) $T_{g,h}(\mathbf{Z})$ is a free $\mathbf{Z}$-module of finite rank satisfying that $T_{g,h}(\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = T_{g,h}(\mathbf{C})$, and that $T_{g,0}(\mathbf{Z}) = \mathbf{Z}$, $T_{g,h}(\mathbf{Z}) = \{0\}$ if $h < 0$. Furthermore, the ring of integral Teichmüller modular forms of degree g :

$$T_g^*(\mathbf{Z}) \stackrel{\text{def}}{=} \bigoplus_{h \geq 0} T_{g,h}(\mathbf{Z})$$

becomes a normal ring which is finitely generated over $\mathbf{Z}$.

(2) For the product θ_g of even theta constants of degree g,

$$N_g \stackrel{\text{def}}{=} \begin{cases} -2^{28} & (g = 3), \\ 2^{2^{g-1}(2^g-1)} & (g \geq 4). \end{cases}$$

Then $\sqrt{\tau^*(\theta_g)/N_g}$ is a **primitive** element of $T_{g,2^{g-3}(2^g+1)}(\mathbf{Z})$, i.e., not congruent to 0 modulo any prime.

(3) $T_3^*(\mathbf{Z})$ is generated by Siegel modular forms over $\mathbf{Z}$ and by $\sqrt{\tau^*(\theta_3)/N_3}$ which is of weight 9, hence is not a Siegel modular form.

Proof. (1) follows from the argument in the proof of Theorem 6.1 (2) replacing

$$\left(\mathcal{A}_g, \ \overline{\mathcal{A}}_g, \ \overline{\lambda} \right) \quad \text{by} \quad \left(\mathcal{M}_g, \ \overline{\mathcal{M}}_g, \ \bigwedge^g \pi_*(\omega_{\mathcal{C}/\overline{\mathcal{M}}_g}) \right),$$

where $\pi : \mathcal{C} \to \overline{\mathcal{M}}_g$ denotes the universal stable curve over Deligne-Mumford's compactification. κ_Δ is used to show that any integral Teichmüller modular form can be extended to a global section on $\overline{\mathcal{M}}_g$.

(2) Let D be the divisor of $\mathcal{M}_g \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ consisting of curves C which have a line bundle L such that $L^{\otimes 2} \cong \Omega_C$ and that $\dim H^0(C, L)$ is positive and even. Then as is shown in [Ty2], $2D$ gives the divisor of $\tau^*(\theta_g)$, and hence a Teichmüller modular form of weight (the weight of θ_g)/2 with divisor D, which exists and is uniquely determined up to constant, is a root of $\tau^*(\theta_g)$ up to constant (see below). Since D is defined over $\overline{\mathbf{Q}}$, a root of $\tau^*(\theta_g)$ times a certain number is defined and primitive over $\mathbf{Z}$. To determine this number, κ_Δ is used as follows: Let A_0, A_Δ, p_{ij} be as in Example 4.1. Then

$$\theta_g(Z) = 2^{2^{g-1}(2^g-1)} \left(\prod_{\substack{(b_1, ..., b_g) \, \in \, \{0, 1/2\}^g \\ \sum_i b_i \, \in \, \mathbf{Z}}} (-1)^{\sum_i b_i} \right) P \cdot \alpha^2,$$

where

$$\alpha \quad : \quad \text{a primitive element of } \mathbf{Z}\left[q_{ij}^{\pm 1} \ (i \neq j) \right] [[q_{11}, ..., q_{gg}]],$$

$$P \quad = \quad \prod_{\substack{(b_1, ..., b_g) \, \in \, \{0, 1/2\}^g \\ \sum_i b_i \, \in \, \mathbf{Z}}} \frac{1}{2} \sum_{S \subset \{1, ..., g\}} (-1)^{\sharp\{k \in S | b_k \neq 0\}} \prod_{i \in S, j \notin S} q_{ij}^{-1/2}$$

$$\Rightarrow \quad \left(\text{the constant term of } P|_{q_{ij}=p_{ij}} \in A_\Delta \right)\big|_{x_1 = x_{-2}, ..., x_g = x_{-1}} = 1.$$

Hence we have (see Exercise 15 below):

$$\sqrt{\text{the constant term of } P|_{q_{ij}=p_{ij}}} \in A_0$$

$$\Rightarrow \quad \sqrt{\theta_g|_{q_{ij}=p_{ij}}} \in \begin{cases} \sqrt{-1} \cdot 2^{27} \cdot A_\Delta & (g = 3), \\ 2^{2^{g-1}(2^g-1)-1} \cdot A_\Delta & (g \geq 4). \end{cases}$$

(3) Recall the result of Igusa [Ig2] that the ideal of $S_3^*(\mathbf{C})$ vanishing on the hyperelliptic locus is generated by θ_3. Since the Torelli map $\mathcal{M}_3 \to \mathcal{A}_3$ is dominant and of degree 2, if we denote ι by the multiplication by -1 on abelian varieties, then

$$\bigoplus_{h: \text{ even}} T_{3,h}(\mathbf{C}) = \{f \in T_3^*(\mathbf{C}) \mid \iota(f) = f\} = S_3^*(\mathbf{C}),$$

$$\bigoplus_{h: \text{ odd}} T_{3,h}(\mathbf{C}) = \{f \in T_3^*(\mathbf{C}) \mid \iota(f) = -f\}.$$

Let f have odd weight. Then by $\iota(f) = -f$, $f = 0$ on the hyperelliptic locus, and hence by Igusa's result, f^2/θ_3 becomes a Siegel modular form. Therefore, $T_3^*(\mathbf{C})$ is generated by $S_3^*(\mathbf{C})$ and $\sqrt{\tau(\theta_3)}$ which implies (3) because $\sqrt{\tau(\theta_3)/N_3}$ is integral and primitive. QED.

Exercise 15. Prove that

$$\left(\prod_{\substack{(b_1, \ldots, b_g) \in \{0, 1/2\}^g \\ \sum_i b_i \in \mathbf{Z}}} (-1)^{\sum_i b_i} \right) = \begin{cases} 1 & (g = 3), \\ -1 & (g \geq 4). \end{cases}$$

TMFs of degree 2. Let k be an algebraically closed field k of characteristic $\neq 2$. Then any proper smooth curve C of genus 2 over k is hyperelliptic, more precisely a base of $H^0(C, \Omega_C)$ gives rise to a morphism $C \to \mathbb{P}_k^1$ of degree 2 ramified at 6 points, and hence

$$\mathcal{M}_2 \otimes_{\mathbf{Z}} k \cong \left\{ (x_1, x_2, x_3 \in \mathbb{P}_k^1 - \{0, 1, \infty\} \mid x_i \neq x_j \ (i \neq j)\right\} / S_6,$$

where each element σ of the symmetric group S_6 degree 6 acts on (x_1, x_2, x_3)'s such as

$$(\sigma(x_1), \sigma(x_2), \sigma(x_3), 0, 1, \infty)$$

is obtained from $\sigma(x_1, x_2, x_3, 0, 1, \infty)$ by some Möbius transformation of $GL_2(k)$. Therefore, $\mathcal{M}_2 \otimes_{\mathbf{Z}} k$ becomes an affine variety, and $T_{2,h}(k) = H^0(\mathcal{M}_2, \lambda^{\otimes h} \otimes_{\mathbf{Z}} k)$ is infinite dimensional. In fact, it is proved in [I3] that the ring

$$T_2^*(\mathbf{Z}) \stackrel{\text{def}}{=} \bigoplus_{h \in \mathbf{Z}} T_{2,h}(\mathbf{Z})$$

of integral Teichmüller modular forms is generated by $\tau^*(S_2^*(\mathbf{Z}))$ and by $2^{12}/(\tau^*(\theta_2))^2$ which is of weight -10.

Construction of TMFs. Assume that $g \geq 3$. Then by results of Mumford [Mu1] and Harer [H1], the **Picard group** of $\mathcal{M}_g$:

$$\mathrm{Pic}(\mathcal{M}_g) \quad \overset{\mathrm{def}}{=} \quad \text{the group of linear equivalence classes of line bundles on } \mathcal{M}_g.$$

is isomorphic to $H^2(\mathcal{M}_g(\mathbf{C}), \mathbf{Z}) \cong H^2(\Pi_g, \mathbf{Z})$ (Π_g denotes the Teichmüller modular group of degree g), and this is free of rank 1 generated by the Hodge line bundle λ. Therefore,

$$D \neq 0 \text{ is an effective divisor on } \mathcal{M}_g \text{ over a subfield } K \text{ of } \mathbf{C}$$
$$\Rightarrow \quad \text{there are } h \in \mathbf{N} \text{ such that } \mathcal{O}_{\mathcal{M}_g}(D) \cong \lambda^{\otimes h}$$
$$\Rightarrow \quad \text{there is } f \in T_{g,h}(K) \text{ such that } (f) = D$$
$$\text{(for the application, see the proof of Theorem 6.3 (2)),}$$

and f is uniquely determined (up to a nonzero constant) by the existence of the Satake-type compactification of $\mathcal{M}_g$. From this method, one can construct Teichmüller modular forms and study their rationality using κ_Δ.

Mumford's isomorphism. We recall Mumford's isomorphism (for $g > 1$) given in 5.3:

$$\det\left(\pi_*\left(\Omega^n_{\mathcal{C}/\mathcal{M}_g}\right)\right) \cong \det\left(\pi_*\left(\Omega_{\mathcal{C}/\mathcal{M}_g}\right)\right)^{\otimes(6n^2-6n+1)} = \lambda^{\otimes(6n^2-6n+1)}.$$

Therefore, by putting $n = 2$,

$$\bigwedge^{3g-3} \pi_*\left(\Omega^2_{\mathcal{C}/\overline{\mathcal{M}}_g}\right) \cong \lambda^{\otimes 13}.$$

In order to express this isomorphism explicitly, we consider the morphism

$$\rho_g : \mathrm{Sym}^2\left(\pi_*\left(\Omega_{\mathcal{C}/\mathcal{M}_g}\right)\right) \ni (s, s') \mapsto s \cdot s' \in \pi_*\left(\Omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}_g}\right)$$

between vector bundles on $\mathcal{M}_g$.

- If $g = 1$, then ρ_1 is an isomorphism and gives

$$\lambda^{\otimes 2} \xrightarrow{\rho_1} \pi_*\left(\Omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}_2}\right) \cong \lambda^{\otimes 14} \quad \Rightarrow \quad \mathcal{O}_{\mathcal{M}_1} \ni 1 \mapsto \pm\Delta(\tau) = q \prod_{n=1}^\infty (1 - q^n)^{24} \in T_{1,12}(\mathbf{Z}).$$

- If $g = 2$, then ρ_2 is an isomorphism and gives

$$\lambda^{\otimes 3} \xrightarrow{\det(\rho_2)} \bigwedge^3 \pi_*\left(\Omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}_2}\right) \cong \lambda^{\otimes 13} \quad \Rightarrow \quad \mathcal{O}_{\mathcal{M}_2} \ni 1 \mapsto \pm\left(\tau^*(\theta_2)/2^6\right)^2 \in T_{2,10}(\mathbf{Z}).$$

- If $g = 3$, then ρ_3 is an isomorphism generically and vanishes on the hyperelliptic locus, hence this gives

$$\lambda^{\otimes 4} \xrightarrow{\det(\rho_3)} \bigwedge^6 \pi_*\left(\Omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}_3}\right) \cong \lambda^{\otimes 13} \quad \Rightarrow \quad \mathcal{O}_{\mathcal{M}_3} \ni 1 \mapsto \pm\sqrt{\tau^*(\theta_3)/N_3} \in T_{3,9}(\mathbf{Z}).$$

Furthermore, [I6] showed the explicit formula of the Mumford isomorphism for any g as an infinite product extending $\Delta(\tau)$.

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List of Exercises

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