Arithmetic Geometry on the Moduli Space of Algebraic Curves

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Abstract

We shall review the following subjects:

- Basic theory on algebraic curves and their moduli space;
- Schottky uniformization theory of Riemann surfaces, and its application to arithmetic geometry on the moduli space of algebraic curves.

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§1. Introduction

1.1. Brief history

- Around 1800∼1830, Gauss(1777–1855), Abel(1802–1829) and Jacobi(1804–1851) showed that the inverse function of the elliptic integral:

$$y = \int dx \Big/ \sqrt{f(x)} \quad (f(x) : \text{a polynomial of degree 4 without multiple root})$$

  is an elliptic function, i.e., a double periodic function of the complex variable y, and they expressed the function as an infinite product and the ratios of theta functions.
  ⇒ complex function theory.

- Riemann(1826–1866) constructed Riemann surfaces from algebraic function fields

$$\mathbf{C}(x, y) \quad (x : \text{ a variable, } y : \text{ finite over } \mathbf{C}(x)),$$

  and solved Jacobi's inverse problem using Abel-Jacobi's theorem and Riemann's theta functions.
  ⇒ complex geometry and algebraic geometry (1857).

- Teichmüller(1913–1943) constructed analytic theory on the moduli of Riemann surfaces.

- Mumford constructed the moduli of algebraic curves as an algebraic variety (1956), and studied this geometry. Further, he and Deligne gave its compactification as the moduli of stable curves (1969).

- String theory provided a strong relationship between physics and the theory of moduli of curves.

- Around 1960∼1970, Shimura constructed arithmetic theory on Shimura models with applications to the rationality on Siegel modular forms, and further Chai and Faltings extended his result to any base ring (1990).

- Grothendieck posed a program to realize geometrically the absolute Galois group as the automorphism group of the profinite fundamental group of the moduli of curves (1984).

1.2. Plan of this lecture

We will review the following subjects with some proof:

- Very classical results on algebraic curves over $\mathbf{C}$ and the associated Riemann surfaces: for example, $\wp$-functions and elliptic curves, differential 1-forms and period integrals, Riemann-Roch's theorem, Abel-Jacobi's theorem and Jacobian varieties, degeneration, Schottky uniformization and the description of forms and periods.

- Rather classical results on moduli and families of algebraic curves: for example, moduli of elliptic curves and higher genus curves, stable curves and their moduli (Deligne-Mumford's compactification), the irreducibility of the moduli, Eisenstein series and Tate curve, Mumford curves;

  and recent results on arithmetic version of Schottky uniformization.

- Recent results on arithmetic geometry of the moduli space of algebraic curves: for example, Fourier expansion of (elliptic and Siegel) modular forms and their rationality, Teichmüller modular forms and the Schottky problem, Mumford's isomorphism, Teichmüller groupoids and their arithmetic geometry, Galois and monodromy representations, Grothendieck-Teichmüller group, mixed Tate motives.

We would like to explain that the classical, but not so familiar Schottky uniformization theory gives an explicit description of forms, periods and degeneration of Riemann surfaces, and that this theory can be extended in the category of arithmetic geometry (unifying complex geometry and formal geometry over $\mathbf{Z}$) with useful applications to studying the following objects:

- automorphic forms called Teichmüller modular forms;

- fundamental groupoids called Teichmüller groupoids;

on the moduli space of algebraic curves.

§2. Algebraic curves and Schottky uniformization

2.1. Algebraic curves and Riemann surfaces

Algebraic curves. **Algebraic varieties** are topological spaces obtained by gluing zero sets of polynomials of multiple variables, and closed subsets of algebraic varieties are defined as zero sets of polynomials (Zariski topology). These examples are

$$\text{the } \textbf{projective } \boldsymbol{n}\textbf{-space } \mathbb{P}^n_k \overset{\text{def}}{=} (k^{n+1} - \{(0, ..., 0)\})/k^\times = \{(x_0 : \cdots : x_{n+1}) = (cx_0 : \cdots : cx_{n+1}) \mid c \neq 0\},$$

and its subsets (called **projective varieties** which are **proper** over k ($\doteq$ compact)) as the zero sets of homogeneous polynomials over an algebraically closed field k.

(algebraic) **curves** $\overset{\text{def}}{=}$ 1-dimensional algebraic varieties

Riemann's correspondence. There exists an equivalence (trinity) of the categories:

(the category of) proper smooth curves over $\mathbf{C}$

take $\mathbf{C}$-rational points $\swarrow$ $\searrow$ take function fields

Riemann surfaces $\overset{\text{def}}{=}$ compact 1-dimensional complex manifolds $\overset{\text{make Riemann surfaces}}{\longleftarrow}$ finite extensions of the rational function field $\mathbf{C}(x)$

Genus. The **genus** of a Riemann surface and the corresponding curve is defined as the number of its holes (**Figure**).

Genus 0 case.

the projective line $\mathbb{P}^1_{\mathbf{C}}$

$\swarrow$ $\searrow$

the Riemann sphere $\mathbb{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$ $\longleftarrow$ $\mathbf{C}(x)$

Genus 1 case. For cubic polynomials $f(x) \in \mathbf{C}[x]$ without multiple root,

$$C_f = \left\{(x_0 : x_1 : x_2) \in \mathbb{P}^2_{\mathbf{C}} \;\middle|\; x_0 x_2^2 = x_0^3 f(x_1/x_0)\right\}$$

$\swarrow$ $\searrow$

complex tori $\mathbf{C}/L$ $\overset{\textbf{Figure}}{\longleftarrow}$ $\mathbf{C}(x, y)$; $y^2 = f(x)$

Here

L is a **lattice** in $\mathbf{C}$, i.e., a sub $\mathbf{Z}$-module of rank 2 such that $L \otimes_{\mathbf{Z}} \mathbf{R} = \mathbf{C}$,

$$E_{2k}(L) \stackrel{\text{def}}{=} \sum_{u \in L - \{0\}} \frac{1}{u^{2k}} \ : \ \text{absolutely convergent series for } k > 1,$$

$$f(x) \stackrel{\text{def}}{=} 4x^3 - 60E_4(L)x - 140E_6(L),$$

$$\wp(z) = \wp_L(z) \stackrel{\text{def}}{=} \frac{1}{z^2} + \sum_{u \in L - \{0\}} \left(\frac{1}{(z-u)^2} - \frac{1}{u^2} \right) \ : \ \text{Weierstrass' } \boldsymbol{\wp}\textbf{-function}$$

$$\Rightarrow \begin{cases} \wp(z) \text{ is absolutely and uniformly convergent on any compact subset of } \mathbf{C} - L, \\ z \mapsto (1 : \wp(z) : \wp'(z)) \text{ gives a biholomorphic map } \mathbf{C}/L \stackrel{\sim}{\to} C_f(\mathbf{C}), \end{cases}$$

and

$$\begin{array}{rl} & \mathbf{C}(x,y) \text{ is a quadratic extension of } \mathbf{C}(x) \text{ defined by } y^2 = f(x) \\ \leftrightarrow & C_f \text{ is a double cover of } \mathbb{P}^1_{\mathbf{C}} \text{ ramified at the 3 roots of } f(x) \text{ and } \infty. \end{array}$$

An **elliptic curve** is a proper smooth curve C of genus 1 and with one marked point x_0. Then C has unique commutative group structure defined algebraically with origin x_0. For example, the above C_f with $(0:0:1)$ is an elliptic curve, and the map $\mathbf{C}/L \stackrel{\sim}{\to} C_f(\mathbf{C})$ is also a group isomorphism which follows from the addition law of $\wp(z)$:

$$\wp(z+w) \ = \ -\wp(z) - \wp(w) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} \right)^2.$$

Exercise 2.1. Show the following Laurent expansion of $\wp(z)$:

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) E_{2n+2}(L) z^{2n} \ \ \text{around } z = 0.$$

Further, using this fact, the periodicity of $\wp(z)$:

$$\wp(z+u) = \wp(z) \ \ (u \in L),$$

and the maximum principle on harmonic functions, prove that

$$\wp'(z)^2 = 4\wp(z)^3 - 60E_4(L)\wp(z) - 140E_6(L)$$
$$\left(\text{i.e., } \wp(z) = x \ \Rightarrow \ z = \int \frac{dx}{\sqrt{4x^3 - 60E_4 x - 140E_6}} \right)$$

and that $E_8(L) = \dfrac{9}{7} E_4(L)^2$.

Genus > 1 case. For a Riemann surface R of genus > 1, by Riemann's mapping theorem, its universal cover is biholomorphic to

$$H_1 \ \stackrel{\text{def}}{=} \ \{\tau \in \mathbf{C} \mid \operatorname{Im}(\tau) > 0\} : \text{the } \textbf{Poincaré upper half plane}.$$

Then we have

$$R \cong H_1/\pi_1(R): \text{ called a } \textbf{Fuchsian model},$$

where the fundamental group $\pi_1(R)$ of R is a cocompact discrete subgroup of

$$PSL_2(\mathbf{R}) \overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{R}) \;\middle|\; ad - bc = 1 \right\} \Big/ \{\pm E_2\}$$

which acts on H_1 by the Möbius transformation:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

(in fact, $PSL_2(\mathbf{R})$ is the group $\mathrm{Aut}(H_1)$ of complex analytic automorphisms of H_1).

<u>**Remark.**</u> Let Γ be a congruence subgroup of $SL_2(\mathbf{Z})$, for example

$$\begin{aligned} SL_2(\mathbf{Z}) &\overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}) \;\middle|\; ad - bc = 1 \right\} \\ \supset \ \Gamma_0(N) &\overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \;\middle|\; c \equiv 0 \bmod (N) \right\} \\ \supset \ \Gamma(N) &\overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \;\middle|\; a - 1 \equiv b \equiv c \equiv d - 1 \equiv 0 \bmod (N) \right\} \\ &: \quad \text{the principal congruence subgroup of } SL_2(\mathbf{Z}) \text{ of level } N. \end{aligned}$$

Then H_1/Γ is a *noncompact* 1-dimensional complex manifold, and becomes compact by adding the set $\mathbb{P}^1(\mathbf{Q})/\Gamma$ of **cusps** of Γ. H_1/Γ and $\big(H_1 \cup \mathbb{P}^1(\mathbf{Q})\big)\big/\,\Gamma$ are called **modular curves**.

<u>2.2. Forms, periods and Jacobians</u>

Let R be a Riemann surface of genus $g \geq 1$. Then its fundamental group $\pi_1(R; x_0)$ with base point $x_0 \in R$ is represented by

$$\left\langle \underbrace{\alpha_1, \beta_1, ..., \alpha_g, \beta_g}_{\text{generators}} \;\middle|\; \underbrace{\prod_{i=1}^{g} \left(\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}\right) = 1}_{\text{relation}} \right\rangle,$$

where the generators α_i, β_i are **canonical**, i.e., closed oriented paths in R with base point x_0 such that α_i, β_i intersect as the x, y-axes and that $(\alpha_i \cup \beta_i) \cap (\alpha_j \cup \beta_j) = \{x_0\}$ if $i \neq j$ (**Figure**). Then

<u>**Theorem 2.1.**</u> (Abel, Jacobi, Riemann, see [Mur])

(1) *The space* $H^0(R, \Omega_R)$ *of* ***holomorphic 1-forms*** *on* R *is* g*-dimensional, and is generated by* $\omega_1, ..., \omega_g$ *satisfying that* $\int_{\alpha_i} \omega_j = \delta_{ij}$.

(2) (Rimeann's period relation) *The* ***period matrix***

$$Z \stackrel{\text{def}}{=} \left(\int_{\beta_i} \omega_j \right)_{1 \le i,j \le g}$$

of $(R; \{\alpha_i, \beta_i\}_{1 \le i \le g})$ *is symmetric, and its imaginary part* $\mathrm{Im}(Z)$ *is positive definite.*

(3) (Abel-Jacobi's theorem) *Let*

$$\mathrm{Cl}^0(R) = \left\{ \sum_i m_i P_i \;\middle|\; \begin{array}{l} m_i \in \mathbf{Z},\ P_i \in R \text{ such that} \\ \deg\left(\sum_i m_i P_i\right) \stackrel{\text{def}}{=} \sum_i m_i = 0 \end{array} \right\} \Big/ \left\{ \sum_{P \in R} \mathrm{ord}_P(f) \cdot P \right\}$$

the ***divisor class group*** *with degree* 0 *of* R*, and let* $\mathbf{C}^g/L$ *be the* g*-dimensional complex torus obtained from the lattice* $L \stackrel{\text{def}}{=} \mathbf{Z}^g + \mathbf{Z}^g \cdot Z$ *in* $\mathbf{C}^g$*. Then the map*

$$\sum_j (P_j - Q_j) \mapsto \left(\sum_j \int_{Q_j}^{P_j} \omega_i \right)_{1 \le i \le g}$$

gives rise to a group isomorphism:

$$\mu : \mathrm{Cl}^0(R) \stackrel{\sim}{\longrightarrow} \mathbf{C}^g/L.$$

Remark. It is clear that $(z_1, ..., z_g) \mapsto \left(\exp(2\pi\sqrt{-1}z_1), ..., \exp(2\pi\sqrt{-1}z_g)\right)$ gives the isomorphism

$$\mathbf{C}^g/L \stackrel{\sim}{\to} (\mathbf{C}^\times)^g \Big/ \left\langle \left(\exp\left(2\pi\sqrt{-1} \int_{\beta_i} \omega_j \right) \right)_{1 \le i \le g} \;\middle|\; 1 \le j \le g \right\rangle,$$

and then

$$\exp\left(2\pi\sqrt{-1} \int_{\beta_i} \omega_j \right) \quad (1 \le i, j \le g)$$

are called the **multiplicative periods**. Let $\mathrm{Pic}^0(R)$ denote the **Picard group** with degree 0 of R which is defined as the group of linear equivalence classes of line bundles with degree 0 over R. Then it is known that

$$\begin{array}{rcl} \mathrm{Cl}^0(R) & \cong & \mathrm{Pic}^0(R) \\ D & \leftrightarrow & \mathcal{O}_R(D);\ \mathcal{O}_R(D)(U) \stackrel{\text{def}}{=} \left\{ f \in \mathcal{O}_R(U) \;\middle|\; \sum_{P \in R} \mathrm{ord}_P(f) \cdot P \ge -D \right\} \end{array}$$

is an **abelian variety**, i.e., a proper (commutative) algebraic group over $\mathbf{C}$, and the isomorphism is also a biholomorphic map. This abelian variety is called the **Jacobian variety** of R (or of the associated curve), and denoted by $\mathrm{Jac}(R)$.

Proof. (1) Since $H_1(R,\mathbf{Z}) = \pi_1(R)/[\pi_1(R),\pi_1(R)]$ has rank $2g$ over $\mathbf{Z}$, $\dim_{\mathbf{C}} H^1(R,\mathbf{C}) = 2g$, and hence by the Hodge decomposition:

$$H^1(R,\mathbf{C}) \;\cong\; H^0(R,\Omega_R) \oplus \overline{H^0(R,\Omega_R)} \quad (\overline{*} : \text{the complex conjugation of } *),$$

we have $\dim_{\mathbf{C}} H^1(R,\Omega_R) = g$.

To prove the remains and (2), (3), first we show a generalized form of Riemann's period relation. Let $\mathcal{P}$ the $4g$ oriented sided polygon obtained from R by cutting the paths α_i, β_i $(1 \le i \le g)$ (**Figure**). Fix $P_0 \in \mathcal{P}$, and for a holomorphic 1-form ϕ on R, define $f(P) = \int_{P_0}^{P} \phi$. Then for a meromorphic 1-form ψ on R whose poles belong to the interior $\mathcal{P}^\circ$ of $\mathcal{P}$ (this condition is satisfied by moving slightly α_i, β_i if necessary), using the function $f^{\pm}$ on the boundary $\partial\mathcal{P}$ of $\mathcal{P}$ defined by

$$\begin{aligned} f^+(P) &\overset{\text{def}}{=} \int_{P_0}^{P} \phi \ \ (P \in \alpha_i \cup \beta_i), \\ f^-(P) &\overset{\text{def}}{=} \int_{P_0}^{P} \phi \ \ (P \in -\alpha_i \cup -\beta_i), \end{aligned}$$

we have

$$\begin{aligned} 2\pi\sqrt{-1} \sum_{P \in \mathcal{P}} \mathrm{Res}_P(f\psi) &= \int_{\partial\mathcal{P}} f\psi \ \ (\text{by the residue theorem}) \\ &= \sum_{i=1}^{g} \left(\int_{\alpha_i} f^+\psi + \int_{-\alpha_i} f^-\psi + \int_{\beta_i} f^+\psi + \int_{-\beta_i} f^-\psi \right) \\ &= \sum_{i=1}^{g} \left(\int_{\alpha_i} (f^+ - f^-)\psi + \int_{\beta_i} (f^+ - f^-)\psi \right) \\ &= \sum_{i=1}^{g} \left(\left(-\int_{\beta_i} \phi \right) \left(\int_{\alpha_i} \psi \right) + \left(\int_{\alpha_i} \phi \right) \left(\int_{\beta_i} \psi \right) \right). \end{aligned}$$

Therefore,

$$2\pi\sqrt{-1} \sum_{P \in \mathcal{P}} \mathrm{Res}_P(f\psi) \;=\; \sum_{i=1}^{g} \left(\left(\int_{\alpha_i} \phi \right) \left(\int_{\beta_i} \psi \right) - \left(\int_{\beta_i} \phi \right) \left(\int_{\alpha_i} \psi \right) \right)$$

which we call the **generalized Riemann's period relation**.

In particular, for two holomorphic 1-forms φ, φ', put $f(P) = \int_{P_0}^{P} \varphi$ $(P \in \mathcal{P}^\circ)$, and put

$$A_i = \int_{\alpha_i} \varphi, \quad A_i' = \int_{\alpha_i} \varphi', \quad B_i = \int_{\beta_i} \varphi, \quad B_i' = \int_{\beta_i} \varphi'.$$

Then by the above,

$$\sum_{i=1}^{g}\left(A_iB_i' - B_iA_i'\right) = 0.$$

Further,

$$\begin{aligned}
\mathrm{Im}\left(\sum_{i=1}^{g}\overline{A_i}B_i\right) &= \frac{1}{2\pi\sqrt{-1}}\sum_{i=1}^{g}\left(\overline{A_i}B_i - \overline{B_i}A_i\right)\\
&= \frac{1}{2\pi\sqrt{-1}}\int_{\mathcal{P}}\overline{f}\varphi = \frac{1}{2\pi\sqrt{-1}}\int_R d\left(\overline{f}\varphi\right)\\
&= \int_R dudv = \int_R\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)dxdy\\
&\quad \left(f = u + \sqrt{-1}v,\ z = x + \sqrt{-1}y : \text{local coordinates}\right)
\end{aligned}$$

is positive if φ is not identically 0. Therefore, any holomorphic 1-form φ with $\int_{\alpha_i}\varphi = 0$ $(1 \le i \le g)$ becomes identically 0, and hence for any base ω_i' $(1 \le i \le g)$ of $H^0(R, \Omega_R)$, $\left(\int_{\alpha_i}\omega_j'\right)_{i,j}$ is a regular matrix. This implies (1).

(2) If $\varphi = \omega_i, \varphi' = \omega_j$, then by the above, $\int_{\beta_i}\omega_j - \int_{\beta_j}\omega_i = 0$, hence Z is symmetric. Further, if $\varphi = \sum_{i=1}^{g} c_i\omega_i \in H^0(R, \Omega_R)$ is not 0, then by the above, $\mathrm{Im}(\overline{\boldsymbol{c}}Z^t\boldsymbol{c}) > 0$ $(\boldsymbol{c} \overset{\text{def}}{=} (c_1, ..., c_g))$ which implies that $\mathrm{Im}(Z)$ is positive definite. This proves (2).

(3) If f is a meromorphic function on R, then by the generalized period relation,

$$\begin{aligned}
&\sum_{P\in R}\left(\mathrm{ord}_P(f)\cdot\int_{P_0}^{P}\omega_j\right)\\
=\ &\sum_{P\in\mathcal{P}}\mathrm{Res}_P\left(\int_{P_0}^{P}\omega_j\cdot\frac{df}{f}\right)\\
=\ &\frac{1}{2\pi\sqrt{-1}}\sum_{i=1}^{g}\left(\left(\int_{\alpha_i}\omega_j\right)\left(\int_{\beta_i}\frac{df}{f}\right) - \left(\int_{\beta_i}\omega_j\right)\left(\int_{\alpha_i}\frac{df}{f}\right)\right)\\
\in\ &L
\end{aligned}$$

because $\int_{\alpha_i} df/f, \int_{\beta_i} df/f \in 2\pi\sqrt{-1}\mathbf{Z}$. Hence the map μ in (3) is well-defined. Next, we show the injectivity of μ. By using **Riemann-Roch's theorem:**

$$\begin{aligned}
&\dim_{\mathbf{C}} H^0(R, \mathcal{O}_R(D)) - \dim_{\mathbf{C}} H^0(R, \Omega_R(-D))\\
=\ &\dim_{\mathbf{C}} H^0(R, \mathcal{O}_R(D)) - \dim_{\mathbf{C}} H^1(R, \mathcal{O}_R(D)) \quad \text{(by Serre's duality)}\\
=\ &\deg(D) + 1 - g,
\end{aligned}$$

we have

$$\dim_{\mathbf{C}} H^0(R, \Omega_R(P_1 + P_2)) = g + 1 = \dim_{\mathbf{C}} H^0(R, \Omega_R) + 1 \quad (P_1, P_2 \in R).$$

Let D be a divisor of degree 0 on R such that $\mu(D) \in L$. Then by the above, there is a meromorphic 1-form ψ on R such that $\sum_{P\in R} \mathrm{Res}_P(\psi) \cdot P = D$. Hence by the period relation,

$$\begin{aligned}
\mu(D) &= \left(\sum_{P\in\mathcal{P}} \mathrm{Res}_P \left(\int_{P_0}^{P} \omega_j \cdot \psi \right) \right)_{1\le j\le g} \\
&= \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{g} \left(\left(\int_{\alpha_i} \omega_j \right) \left(\int_{\beta_i} \psi \right) - \left(\int_{\beta_i} \omega_j \right) \left(\int_{\alpha_i} \psi \right) \right)_{1\le j\le g} \\
&\in L = \left\{ \sum_{i=1}^{g} \left(m_i \int_{\alpha_i} \omega_j - l_i \int_{\beta_i} \omega_j \right)_{1\le j\le g} \;\middle|\; m_i, l_i \in \mathbf{Z} \right\}.
\end{aligned}$$

Therefore, there are integers m_i, l_i $(1 \le i \le g)$ such that

$$\sum_{i=1}^{g} \left(\left(\int_{\beta_i} \psi - (2\pi\sqrt{-1})m_i \right) \left(\int_{\alpha_i} \omega_j \right) - \left(\int_{\alpha_i} \psi - (2\pi\sqrt{-1})l_i \right) \left(\int_{\beta_i} \omega_j \right) \right) = 0$$

for any $1 \le j \le g$. By (1), the orthogonal subspace of $\mathbf{C}^{2g}$ to

$$\left(\int_{\alpha_1} \omega_j, \cdots, \int_{\alpha_g} \omega_j, \int_{\beta_1} \omega_j, \cdots, \int_{\beta_g} \omega_j \right) \quad (1 \le j \le g)$$

has dimension g, and by the period relation, this is generated by

$$\left(\int_{\beta_1} \omega_j, \cdots, \int_{\beta_g} \omega_j, -\int_{\alpha_1} \omega_j, \cdots, -\int_{\alpha_g} \omega_j \right) \quad (1 \le j \le g).$$

Hence there are $b_1, ..., b_g \in \mathbf{C}$ such that

$$\int_{\alpha_i} \psi - (2\pi\sqrt{-1})l_i = \sum_{j=1}^{g} b_j \int_{\alpha_i} \omega_j, \quad \int_{\beta_i} \psi - (2\pi\sqrt{-1})m_i = \sum_{j=1}^{g} b_j \int_{\beta_i} \omega_j,$$

and then

$$f = \exp\left(\int_{P_0}^{P} \left(\psi - \sum_{j=1}^{g} b_j \omega_j \right) \right)$$

is a meromorphic function on R such that $\sum_{P\in R} \mathrm{ord}_P(f) \cdot P = D$. This implies the injectivity of μ. Finally, we show that the surjectivity of μ. Let φ_1 be a nonzero holomorphic 1-form, and Q_1 be a point on R at which φ_1 does not vanish. Then by Riemann-Roch's theorem,

$$\dim_{\mathbf{C}} H^0(R, \Omega_R(-Q_1)) = \dim_{\mathbf{C}} H^0(R, \mathcal{O}_R(Q_1)) + g - 2 = g - 1,$$

and hence there are nonzero $\varphi_2 \in H^0(R, \Omega_R(-Q_1))$ and $Q_2 \in R$ at which φ_2 does not vanish. By repeating this process, one take a base $\varphi_1, ..., \varphi_g$ of $H^0(R, \Omega_R)$ and $Q_1, ..., Q_g \in R$ such that $\varphi_{i+1}, ..., \varphi_g$, but not φ_i vanish at Q_i $(1 \leq i \leq g)$. Therefore, for $P_1, ..., P_g$ in neighborhoods of $Q_1, ..., Q_g$ respectively, the jacobian of

$$(P_1, ..., P_g) \longmapsto \left(\sum_{i=1}^{g} \int_{Q_i}^{P_i} \varphi_1, ..., \sum_{i=1}^{g} \int_{Q_i}^{P_i} \varphi_g \right)$$

is nonzero at $(Q_1, ..., Q_g)$, and hence by the implicit function theorem, the linear map μ is locally biholomorphic. This implies that μ is surjective. QED.

Exercise 2.2. Fix $P_0 \in R$. Then for each $P \in R$, prove that there is a unique meromorphic 1-form $w_P = w_P(z)$ on R such that

- w_P is holomorphic except $z = P, P_0$;
- w_P has simple poles at $z = P, P_0$ with residues $1, -1$ respectively;
- $\displaystyle\int_{\alpha_i} w_P = 0.$

Further, using the generalized Riemann's period relation, prove that

$$d\left(\int_{\beta_i} w_z \right) = 2\pi\sqrt{-1}\omega_i(z).$$

Example 2.1. If $R = \mathbf{C}/L$: genus 1; $L = \mathbf{Z} + \mathbf{Z}\tau$ $(\mathrm{Im}(\tau) > 0)$, then

$$H^0(R, \Omega_R) = \mathbf{C}dz, \quad Z = \int_0^\tau dz = \tau.$$

Schottky unifomization gives explicit description of forms and periods

2.3. Degeneration of Riemann surfaces

Genus 1 case. If $f(x)$(: degree 3, without multiple root) becomes $a(x-\alpha)^2(x-\beta)$ $(a \neq 0, \alpha \neq \beta)$, then the complex torus $C_f(\mathbf{C})$ degenerates to a singular space obtained by identifying 2-points on the Riemann sphere (**Figure**).

For example, for $f(x) = (x^2 - \varepsilon^2)(x+1)$,

$$\begin{aligned} & y^2 = f(x) \Leftrightarrow (\sqrt{x^2+x^3}+y)(\sqrt{x^2+x^3}-y) = \varepsilon^2(1+x) \\ \overset{\varepsilon \to 0}{\longrightarrow} \ & (\sqrt{x^2+x^3}+y)(\sqrt{x^2+x^3}-y) = 0 \ \text{ around } (x,y) = (0,0), \end{aligned}$$

where $\sqrt{x^2+x^3} = \sum_{k=0}^{\infty} \begin{pmatrix} 1/2 \\ k \end{pmatrix} x^{k+1}$.

Local degeneration. For a complex number ε such that $0 < |\varepsilon| < 1$, let D be the union of the two annular domains:

$$U = \{x \in \mathbf{C} \mid |\varepsilon| < |x| < 1\}, \quad V = \{y \in \mathbf{C} \mid |\varepsilon| < |y| < 1\}$$

by the relation $xy = \varepsilon$. Then under $\varepsilon \to 0$, D becomes the union of the 2 disks

$$\{x \in \mathbf{C} \mid |x| < 1\}, \quad \{y \in \mathbf{C} \mid |y| < 1\}$$

identifying $x = 0$ and $y = 0$.

Ordinary double points. For a point P on a curve C,

$$\begin{aligned} & P \text{ is an } \textbf{ordinary double point} \text{ (or } \textbf{node}\text{)} \\ \overset{\text{def}}{\Longleftrightarrow} & \left\{ \begin{array}{l} \text{the local equation around } P \in C \text{ is given by } xy = 0 \\ \text{for some formal coordinates } x, y \end{array} \right. \\ \Longleftrightarrow & \ P \text{ is a point of multiplicity 2 with distinct tangent directions} \end{aligned}$$

2.4. Schottky uniformization

Schottky uniformization is to construct Riemann surfaces of genus g from a $2g$ holed Riemann sphere by identifying these holes in pairs (**Figure**). More precisely, let

$$PGL_2(\mathbf{C}) \overset{\text{def}}{=} GL_2(\mathbf{C})/\,\mathbf{C}^{\times}(\cdot E_2)$$

which acts on $\mathbb{P}^1(\mathbf{C})$ by the Möbius transformation, and let

$$\begin{aligned} & D_{\pm 1}, ..., D_{\pm g} \subset \mathbb{P}^1(\mathbf{C}) : \text{ disjoint closed domains bounded by Jordan curves } \partial D_i, \\ & \gamma_1, ..., \gamma_g \in PGL_2(\mathbf{C}) \text{ such that } \gamma_i(\mathbb{P}^1(\mathbf{C}) - D_{-i}) = \text{the interior } D_i^{\circ} \text{ of } D_i, \\ & \Gamma \overset{\text{def}}{=} \langle \gamma_1, ..., \gamma_g \rangle : \text{ the subgroup of } PGL_2(\mathbf{C}) \text{ generated by } \gamma_1, ..., \gamma_g, \\ & \Omega_\Gamma \overset{\text{def}}{=} \bigcup_{\gamma \in \Gamma} \gamma \left(\mathbb{P}^1(\mathbf{C}) - \bigcup_{i=1}^{g} (D_i^{\circ} \cup D_{-i}^{\circ}) \right). \end{aligned}$$

Then the Riemann surface

$$\begin{aligned} R_\Gamma & \overset{\text{def}}{=} \left(\mathbb{P}^1(\mathbf{C}) - \bigcup_{i=1}^{g} (D_i^{\circ} \cup D_{-i}^{\circ}) \right) \Big/ \partial D_i \overset{\gamma_i}{\sim} \partial D_{-i} \ (\because \text{ gluing by } \gamma_i) \\ & = \Omega_\Gamma / \Gamma \end{aligned}$$

is called (Schottky) uniformized by the **Schottky group** Γ. It is known that any Riemann surface can be Schottky uniformized. Counterclockwise oriented ∂D_i and oriented paths from $w_i \in \partial D_{-i}$ to $\gamma_i(w_i) \in \partial D_i$ $(1 \leq i \leq g)$ become canonical generators, and we denote them by α_i, β_i respectively (**Figure**).

Exercise 2.3. Prove that Γ is a free group with generators $\gamma_1, ..., \gamma_g$, and that the action of Γ on Ω_Γ is free and properly discontinuous. Further, prove that each γ_i $(1 \leq i \leq g)$ is uniquely represented by

$$\gamma_i = \begin{pmatrix} t_i & t_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & s_i \end{pmatrix} \begin{pmatrix} t_i & t_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \mod(\mathbf{C}^\times),$$

where $t_i \in D_i^\circ$, $t_{-i} \in D_{-i}^\circ$ and $|s_i| < 1$ (hence γ_i is hyperbolic (or loxodromic)), and that

$$t_{\pm i} \;=\; \lim_{n\to\infty} \gamma_i^{\pm n}(z) \;\; (z \in \Omega_\Gamma).$$

t_i, t_{-i} are called the **attractive**, **repulsive** fixed point of γ_i respectively, and s_i is called the **multiplier** of γ_i.

Theorem 2.2. (Schottky [S]) *Assume that* $\infty \in \Omega_\Gamma$ *and that* $\sum_{\gamma\in\Gamma} |\gamma'(z)|$ *converges uniformly on any compact subset of*

$$\Omega_\Gamma - \bigcup_{\gamma\in\Gamma} \gamma(\infty).$$

Then we have

(1) *For* $n \geq 1$ *and a point* $p \in \Omega_\Gamma - \bigcup_{\gamma\in\Gamma} \gamma(\infty)$,

$$w_{n,p}(z) \;\stackrel{\text{def}}{=}\; \sum_{\gamma\in\Gamma} \frac{d\gamma(z)}{(\gamma(z)-p)^n} = \sum_{\gamma\in\Gamma} \frac{\gamma'(z)}{(\gamma(z)-p)^n} dz$$

becomes a meromorphic 1-form on R_Γ. *If* $n > 1$, *then* $w_{n,p}$ *is of the 2-nd kind, i.e., has only poles of order* n *at the point* $\overline{p}$ *on* R_Γ *induced from* p, *and if* $n = 1$, *then* $w_{n,p}$ *is of the 3-rd kind, i.e., has only simple poles at* $\overline{p}, \overline{\infty}$. *Furthermore, for* $n \geq 0$,

$$\sum_{\gamma\in\Gamma} \gamma(z)^n d\gamma(z) = \sum_{\gamma\in\Gamma} \gamma(z)^n \cdot \gamma'(z) dz$$

becomes a meromorphic 1-form on R_Γ *which has only pole of order* $n+2$ *at* $\overline{\infty}$.

(2) *For* $i = 1, ..., g$,

$$\omega_i(z) \;=\; \frac{1}{2\pi\sqrt{-1}} \sum_{\gamma\in\Gamma/\langle\gamma_i\rangle} \left(\frac{1}{z-\gamma(t_i)} - \frac{1}{z-\gamma(t_{-i})} \right) dz$$

give a base of $H^0(R_\Gamma, \Omega_{R_\Gamma})$ *satisfying that* $\displaystyle\int_{\alpha_i} \omega_j = \delta_{ij}$.

(3) *For* $1 \leq i, j \leq g$ *and* $\gamma \in \Gamma$, *put*

$$\psi_{ij}(\gamma) = \begin{cases} s_i & (\text{if } i = j \text{ and } \gamma \in \langle\gamma_i\rangle), \\ \dfrac{(t_i - \gamma(t_j))(t_{-i} - \gamma(t_{-j}))}{(t_i - \gamma(t_{-j}))(t_{-i} - \gamma(t_j))} & (\text{otherwise}). \end{cases}$$

Then we have

$$\exp\left(2\pi\sqrt{-1}z_{ij}\right) = \prod_{\gamma\in\langle\gamma_i\rangle\backslash\Gamma/\langle\gamma_j\rangle} \psi_{ij}(\gamma),$$

where $Z = (z_{ij})_{i,j}$ is the period matrix of $(R_\Gamma; (\alpha_i, \beta_i)_{1\le i\le g})$.

Proof. The assertion (1) is evident except the convergence of $w_{n,p}(z)$ which follows from the assumption and that the action of Γ on Ω_Γ is properly discontinuous. Further, $w_{1,p}(z)$ has simple poles at $\overline{p}, \overline{\infty}$ with residues $1, -1$ respectively, and satisfies that $\int_{\alpha_i} w_{1,p} = 0$ $(1 \le i \le g)$. Then by Exercise 2.2,

$$\begin{aligned}
2\pi\sqrt{-1}\omega_i(z) &= d\left(\int_{\zeta_i}^{\gamma_i(\zeta_i)} \sum_{\gamma\in\Gamma} \frac{d\gamma(\zeta)}{\gamma(\zeta)-z}\right);\ \zeta_i \text{ is a point on } \partial D_{-i} \\
&= \left(\sum_{\gamma\in\Gamma} \log\left(\frac{(\gamma\gamma_i)(\zeta_i)-z}{\gamma(\zeta_i)-z}\right)\right) dz \\
&= \sum_{\gamma\in\Gamma}\left(\frac{1}{z-(\gamma\gamma_i)(w_i)} - \frac{1}{z-\gamma(w_i)}\right) dz \\
&= \sum_{\gamma\in\Gamma/\langle\gamma_i\rangle}\sum_{n\in\mathbf{Z}}\left(\frac{1}{z-(\gamma\gamma_i^{n+1})(w_i)} - \frac{1}{z-(\gamma\gamma_i^{n})(w_i)}\right) dz,
\end{aligned}$$

and since $t_{\pm i} = \lim_{n\to\infty} \gamma_i^{\pm n}(w_i) \in D_{\pm i}^{\circ}$ (Exercise 2.3), we have

$$\omega_i(z) = \frac{1}{2\pi\sqrt{-1}} \sum_{\gamma\in\Gamma/\langle\gamma_i\rangle}\left(\frac{1}{z-\gamma(t_i)} - \frac{1}{z-\gamma(t_{-i})}\right) dz,$$

which proves (2). QED.

Exercise 2.4. Prove that $\int_{\alpha_i} w_{1,p} = 0$ $(1 \le i \le g)$, and check that ω_i is Γ-invariant and $\int_{\alpha_i} \omega_j = \delta_{ij}$.

Exercise 2.5. Prove (3) of Theorem 2.2.

Proposition. *Assume that $\Omega_\Gamma \ni \infty$, and that $t_{\pm i}$ are fixed and s_i are sufficiently small, then the assumption in Theorem 2.2 is satisfied.*

Proof. For 2 disks $D_i, D_j \subset \mathbf{C}$ with radius r_i, r_j respectively, put

$$\begin{aligned}
\rho_{i,j} &: \text{ the distance between the centers of } D_i \text{ and } D_j, \\
K_{i,j} &= \frac{(r_i^2 + r_j^2 - \rho_{i,j}^2)^2}{4r_i^2 r_j^2} - 1 \ge 0, \\
L_{i,j} &= \frac{1}{\sqrt{1+K_{i,j}} + \sqrt{K_{i,j}}} \le 1.
\end{aligned}$$

Then $K_{i,j}$ and $L_{i,j}$ are invariant under any Möbius transformation, and $r_i \leq L_{i,j} \cdot r_j$ if $D_i \subset D_j$. Under the assumption, one can take disks $D_{\pm 1}, ..., D_{\pm g}$ such that the sum of $L_{i,j}$ $(i, j \in \{\pm 1, ..., \pm g\}, i \neq j)$ is smaller than 1. Hence by the above, there is a positive constant C such that if $\gamma = \prod_{s=1}^{l} \gamma_{k(s)} \in \Gamma$ is expressed as

$$\begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \bmod(\mathbf{C}^\times); \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in SL_2(\mathbf{C}),$$

then

$$\frac{1}{|c_\gamma|^2} \leq C \cdot \prod_{s=1}^{l-1} L_{-k(s),k(s+1)}.$$

Therefore,

$$\sum_{\gamma \in \Gamma - \{1\}} \frac{1}{|c_\gamma|^2} \leq C \cdot \sum_{m=0}^{\infty} \left(\sum_{i \neq j} L_{i,j} \right)^m < \infty,$$

and hence

$$\sum_{\gamma \in \Gamma} |\gamma'(z)| \leq 1 + \frac{1}{d(z)^2} \sum_{\gamma \in \Gamma - \{1\}} \frac{1}{|c_\gamma|^2}$$

satisfies the condition since $d(z) \overset{\text{def}}{=} \min\{|z - \gamma^{-1}(\infty)|; \gamma \in \Gamma\} > 0$ is bounded on any compact subset outside $\bigcup_{\gamma \in \Gamma} \gamma(\infty)$. QED

Remark. Schottky [S] gives a (more geometric) convergence condition on $\sum_{\gamma \in \Gamma} |\gamma'(z)|$ as follows: all $\partial D_{\pm i}$ can be taken as circles (in this case, Γ is called classical) and there are $2g-3$ circles $C_1, ..., C_{2g-3}$ in $F = \mathbb{P}^1(\mathbf{C}) - \bigcup_{i=1}^{g} \left(D_i^\circ \cup D_{-i}^\circ \right)$ satisfying that

- $C_1, ..., C_{2g-3}$, $\partial D_{\pm 1}, ..., \partial D_{\pm g}$ are mutually disjoint;
- $C_1, ..., C_{2g-3}$ divide F into $2g-2$ domains $R_1, ..., R_{2g-2}$;
- each R_i has exactly three boundary circles.

Variation of forms and periods. Let $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$ be a Schottky group of rank g as above, and put $\Gamma' = \langle \gamma_1, ..., \gamma_{g-1} \rangle$ which is a Schottky group of rank $g-1$. If the multiplier

$$\begin{aligned} s_g &= \frac{\gamma_g(z) - t_g}{z - t_g} \cdot \frac{z - t_{-g}}{\gamma_g(z) - t_{-g}} \\ &: \text{ the product of local coordinates around } t_g, t_{-g} \text{ respectively} \end{aligned}$$

of γ_g tends to 0, then

- $R_\Gamma \longrightarrow \begin{cases} \text{the singular curve } R'_{\Gamma'} \text{ with unique singular (ordinary double) point} \\ \text{obtained by identifying 2 points } t_g, t_{-g} \in R_{\Gamma'}; \end{cases}$

- $2\pi\sqrt{-1}\,\omega_i(z) = \sum_{\gamma\in\Gamma/\langle\gamma_i\rangle} \left(\frac{1}{z-\gamma(t_i)} - \frac{1}{z-\gamma(t_{-i})}\right) dz \in H^0\left(R_\Gamma, \Omega_{R_\Gamma}\right)$

$$\longrightarrow \begin{cases} \displaystyle\sum_{\gamma\in\Gamma'/\langle\gamma_i\rangle} \left(\frac{1}{z-\gamma(t_i)} - \frac{1}{z-\gamma(t_{-i})}\right) dz & (i<g), \\ \displaystyle\left(\frac{1}{z-t_g} - \frac{1}{z-t_{-g}}\right) dz + \cdots & (i=g) \end{cases}$$

which has a pole at the ordinary double point $t_g = t_{-g}$ on $R'_{\Gamma'}$ if $i=g$;

- (Fay's formula [Fa]) $p_{ij} \longrightarrow \begin{cases} \text{the multiplicative periods of } R_{\Gamma'} & (i,j<g), \\ 0 & (i=j=g). \end{cases}$

Therefore, on the complex geometry of $R'_{\Gamma'}$, it is natural to replace the sheaf of holomorphic 1-forms on $R'_{\Gamma'}$ by that of 1-forms η on $R_{\Gamma'}$ holomorphic except for simple poles at t_g, t_{-g} satisfying that $\mathrm{Res}_{t_g}(\eta) + \mathrm{Res}_{t_{-g}}(\eta) = 0$ (see 3.2 below).

Remark. We can obtain variational formula under other degenerations (see [I3]).

§3. Moduli space of algebraic curves

3.1. Construction of moduli

A **scheme** is a locally ringed space which is locally given by the affine scheme:

$$\mathrm{Spec}(A) \overset{\text{def}}{=} \{\text{prime ideals of } A\} \ni \mathfrak{p} \mapsto A_{\mathfrak{p}} \overset{\text{def}}{=} \{a/s \mid a \in A,\ s \in A - \mathfrak{p}\}$$

associated with a commutative ring A with unit 1 (the category of affine schemes is contravariantly equivalent to that of commutative rings with unit 1). A scheme X over a scheme S is a scheme with morphism $X \to S$.

The **moduli space of curves** is a space representing

the isomorphism classes of curves.

More precisely, if $\mathcal{M}_g$ is a *fine* moduli of curves of genus g, then

$$\begin{aligned} \mathcal{M}_g(S) &\overset{\text{def}}{=} \{\text{morphisms from } S \text{ to } \mathcal{M}_g\} \quad (S : \text{schemes}) \\ &\overset{\text{functorial}}{\cong} \{\text{isomorphism classes of curves over } S \text{ of genus } g\} \end{aligned}$$

Caution! There is no fine moduli as an scheme since there are curves with nontrivial automorphism (for example, **hyperelliptic curves** defined by $y^2 = f(x)$ has a nontrivial automorphism $x \mapsto x$, $y \mapsto -y$).

Solutions.

(S1) Construct the fine moduli as an scheme by considering additional structures on curves.

(S2) Taking the categorical quotient of the above fine moduli, construct the fine moduli as an **algebraic stack**, the scheme-theoretic analog of **orbifolds**, which is represented as

$$[U/R] : \text{ the quotient of } U \text{ by } R,$$

where U, R are schemes with etale morphisms $s, t : R \to U$ and a morphism $\mu : R \times_{U,t,s} R \to R$ such that $(s,t) : R \to U \times U$ is finite and s, t, μ form a groupoid. For a scheme S, $[U/R](S) = \mathrm{Hom}(S, U/R)$ is the category given by

$$\begin{aligned} \mathrm{Ob}\left([U/R](S)\right) &\overset{\text{def}}{=} \mathrm{Hom}(S, U), \\ \mathrm{Mor}\left([U/R](S)\right) &\overset{\text{def}}{=} \left\{\alpha \in \mathrm{Hom}(S, R) \text{ giving } s \circ \alpha \overset{\sim}{\to} t \circ \alpha\right\}, \end{aligned}$$

and R gives the equivalence relation by μ.

(S3) Taking the geometric quotient of the above fine moduli, construct the *coarse* moduli as an scheme.

Moduli of elliptic curves.

- **(S1)** If E is an elliptic curve over $\mathbf{C}$, and ι is an isomorphism $\mathbf{Z}^{\oplus 2} \xrightarrow{\sim} H_1(E, \mathbf{Z})$ such that ι is *canonical*, i.e., $\iota(\boldsymbol{e}_1), \iota(\boldsymbol{e}_2)$ intersects as the x, y-axes, then the ratio
$$\left(\int_{\iota(\boldsymbol{e}_2)} \omega\right) \Big/ \left(\int_{\iota(\boldsymbol{e}_1)} \omega\right)$$
is independent of $\omega \in H^1(E, \Omega_E) - \{0\}$ and belongs to the Poincaré upper half plane H_1. Therefore, by the correspondence:
$$H_1 \ni \tau \;\leftrightarrow\; (\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau);\; \iota(\boldsymbol{e}_1) = 1, \iota(\boldsymbol{e}_2) = \tau)\,,$$
H_1 becomes the fine moduli space of elliptic curves E over $\mathbf{C}$ with canonical isomorphism $\mathbf{Z}^{\oplus 2} \xrightarrow{\sim} H_1(E, \mathbf{Z})$.

  (S2) By (S1), the fine moduli stack of elliptic curves over $\mathbf{C}$ is given by the complex analytic stack, i.e. **orbifold**
$$[H_1/SL_2(\mathbf{Z})]\,.$$

  (S3) Since

$$\begin{aligned} & \text{the elliptic curves } y_i^2 = 4x_i^3 - \alpha_i x_i - \beta_i \ (i = 1, 2) \text{ over } \mathbf{C} \text{ are isomorphic} \\ \Leftrightarrow\ & \text{there are } a, b, c, d, e \in \mathbf{C} \text{ with } a, c \neq 0 \text{ such that} \\ & \begin{cases} x_2 = ax_1 + b : \text{ order} \geq -2 \text{ at the origin,} \\ y_2 = cy_1 + dx_1 + e : \text{ order} \geq -3 \text{ at the origin} \end{cases} \\ \Leftrightarrow\ & \text{there are } a, c \in \mathbf{C}^\times \text{ such that } a^3 = c^2, x_2 = ax_1, y_2 = cy_1 \\ \Leftrightarrow\ & \text{the } \boldsymbol{j}\textbf{-invariants } \frac{\alpha_i^3}{\alpha_i^3 - 27\beta_i^2} \text{ of } y_i^2 = 4x_i^3 - \alpha_i x_i - \beta_i \ (i = 1, 2) \text{ are equal} \\ & \big(\text{note that } \alpha_i^3 - 27\beta_i^2 \neq 0\big)\,, \end{aligned}$$

  the coarse moduli scheme of elliptic curves over $\mathbf{C}$ becomes the affine line over $\mathbf{C}$, and the j-function
$$\begin{aligned} j(\tau) &\overset{\text{def}}{=} \frac{(60E_4(\mathbf{Z} + \mathbf{Z}\tau))^3}{(60E_4(\mathbf{Z} + \mathbf{Z}\tau))^3 - 27(140E_6(\mathbf{Z} + \mathbf{Z}\tau))^2} \\ &= \frac{1}{1728}\left(\frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots\right) \quad \left(q \overset{\text{def}}{=} e^{2\pi\sqrt{-1}\tau}\right) \end{aligned}$$
  gives a biholomorphic map from the geometric quotient $H_1/SL_2(\mathbf{Z})$ onto $\mathbf{C}$.

- **(S1)** For complex numbers $\mu \neq 1,\ \zeta_3 = e^{2\pi\sqrt{-1}/3},\ \zeta_3^2$, put
$$\begin{aligned} E(\mu) &\overset{\text{def}}{=} \big\{(x_0 : x_1 : x_2) \in \mathbb{P}^1(\mathbf{C}) \mid x_0^3 + x_1^3 + x_2^3 = 3\mu x_0 x_1 x_2\big\} : \textbf{ Hesse's cubic} \\ &: \ \text{an elliptic curve over } \mathbf{C} \text{ with origin } (1 : -1 : 0) \text{ containing} \\ &\quad\ \text{3-division points } (1 : -\beta : 0), (0 : 1 : -\beta), (-\beta : 0 : 1) \ \big(\beta = 1, \zeta_3, \zeta_3^2\big) \end{aligned}$$

Then $\mu \mapsto (E(\mu)$ with the 3-division points) gives a bijection:

$$\begin{aligned} \mathbf{C} - \left\{1, \zeta_3, \zeta_3^2\right\} &\overset{\sim}{\to} \left\{ \begin{array}{l} \text{isomorphism classes of elliptic curves over } \mathbf{C} \\ \text{with symplectic level 3 structure} \\ (\mathbf{Z}/3\mathbf{Z})^{\oplus 2} \overset{\sim}{\to} E[3] \overset{\text{def}}{=} \{P \in E \mid 3P = 0\} \end{array} \right\} \\ &\cong H_1/\Gamma(3), \end{aligned}$$

where $\Gamma(3)$ denotes the principal congruence subgroup of $SL_2(\mathbf{Z})$ of level 3. Therefore, $\mathbf{C} - \left\{1, \zeta_3, \zeta_3^2\right\}$ has a natural model over $\mathbf{Z}[1/3, \zeta_3]$ as the fine moduli scheme of elliptic curves with level 3 structure, and this can be compactified to $\mathbb{P}^1$ by adding the 4 points $1, \zeta_3, \zeta_3^2, \infty$ which correspond degenerate curves. I. Nakamura [Na] gave this higher dimensional version, i.e., constructed a compactification of the moduli of principally polarized abelian varieties with level structure as an moduli space.

(S2) The fine moduli stack over $\mathbf{Z}[1/3, \zeta_3]$ is given by the quotient stack of the above model in (S1) by $SL_2(\mathbf{Z}/3\mathbf{Z})$.

- **(S1)** If E is an elliptic curve over a scheme S with 0-section $e : S \to E$, then

$$H^0(E, \mathcal{O}(3 \cdot e(S))) : \text{ the space of generalized elliptic functions}$$

defines an embedding $E \hookrightarrow \mathbb{P}^2_S$. Therefore, by the theory of Hilbert schemes, there is a fine moduli scheme $\mathcal{H}_1$ over $\mathbf{Z}$ classifying elliptic curves with embedding into $\mathbb{P}^2_S$ as above.

(S2) The fine moduli stack over $\mathbf{Z}$ of elliptic curves is given by the quotient stack

$$\left[\mathcal{H}_1/\mathrm{Aut}(\mathbb{P}^2)\right] = \left[\mathcal{H}_1/PGL_3\right]; \quad PGL_n \overset{\text{def}}{=} GL_n/\mathbf{G}_m.$$

Exercise 3.1. Prove that for $\tau \in H_1$,

$$\begin{aligned} &\{\gamma \in SL_2(\mathbf{Z}) \mid \gamma(\tau) = \tau\} \\ = &\left\{ \begin{array}{ll} \left\langle \rho \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \rho^{-1} \right\rangle & : \text{order } 6 \quad (\text{if } \exists \rho \in SL_2(\mathbf{Z}) \text{ such that } \tau = \rho(\zeta_3)), \\ \left\langle \rho \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rho^{-1} \right\rangle & : \text{order } 4 \quad (\text{if } \exists \rho \in SL_2(\mathbf{Z}) \text{ such that } \tau = \rho(\sqrt{-1})), \\ \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle & : \text{order } 2 \quad (\text{otherwise}). \end{array} \right. \end{aligned}$$

Construction of moduli for genus > 1. There are 3 approaches using

1. Teichmüller theory: Fix a Riemann surface R_0 of genus $g > 1$. Then the **Teichmüller space** of degree g is defined by

$$T_g \overset{\text{def}}{=} \left\{ (R,h) \;\middle|\; \begin{array}{l} R : \text{Riemann surfaces of genus } g \\ h : \text{orientation preserving diffeomorphisms } R_0 \to R \end{array} \right\} \Big/ \sim$$

$$; \quad (R,h) \sim (R',h') \overset{\text{def}}{\Longleftrightarrow} h' \circ h^{-1} \text{ is homotopic to a biholomorphic map,}$$

and the **Teichmüller modular group** or **mapping class group** of degree g is defined by

$$\Pi_g \overset{\text{def}}{=} \{\text{homotopy classes of orientation preserving diffeomorphisms } R_0 \to R_0\}$$

which acts on T_g as $\mu(R,h) = (R, h \circ \mu)$ ($\mu \in \Pi_g$) properly discontinuously. Then the quotient orbifold $[T_g/\Pi_g]$ exists and becomes the moduli space of Riemann surfaces of genus g. Teichmüller proved that T_g is homeomorphic to $\mathbf{R}^{6g-6}$ and becomes naturally a complex manifold of dimension $3g-3$ by using the theory of quasiconformal maps (see [IT]). Since T_g is connected and simply connected,

$$\pi_1([T_g/\Pi_g]) \cong \Pi_g,$$

and these are canonically isomorphic to

$$\mathrm{Aut}^+(\pi_1(R_0)) \big/ \mathrm{Inn}(\pi_1(R_0)),$$

where $\mathrm{Aut}^+(\pi_1(R_0))$ denotes the automorphism group of $\pi_1(R_0)$ preserving the (alternating and bilinear) intersection form on $H_1(R_0, \mathbf{Z}) = \pi_1(R_0)/[\pi_1(R_0), \pi_1(R_0)]$, and $\mathrm{Inn}(\pi_1(R_0))$ denotes the inner automorphism group.

Caution! Is is shown by Royden that if $g > 1$, then $\mathrm{Aut}(T_g) = \Pi_g$, and hence the T_g is not a homogeneous space. Therefore, one cannot regard the Teichmüller modular group as a discrete subgroup of a Lie group.

2. Moduli theory of abelian varieties: A **principally polarized abelian variety** (A, φ) is a pair of an abelian variety A, i.e., a proper (commutative) algebraic group and an isomorphism $A \to \widehat{A}$ (: the dual abelian variety of A) induced from an ample divisor on A. There exists a moduli space $\mathcal{A}_g$ of principally polarized g-dimensional abelian varieties, and

$$\mathcal{A}_g(\mathbf{C}) \cong [H_g / Sp_{2g}(\mathbf{Z})].$$

Here

$$H_g \overset{\text{def}}{=} \{Z \in M_g(\mathbf{C}) \mid Z : \text{symmetric}, \ \mathrm{Im}(Z) > 0\}$$

: the **Siegel upper half space** of degree g,

$$Sp_{2g}(\mathbf{Z}) \stackrel{\text{def}}{=} \left\{ G \in M_{2g}(\mathbf{Z}) \;\middle|\; G \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix} {}^tG = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix} \right\}$$

: the **integral symplectic group** of degree g over $\mathbf{Z}$

acts on H_g as $Z \mapsto (AZ+B)(CZ+D)^{-1}$ for $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

and $Z/\sim\ \in H_g/Sp_{2g}(\mathbf{Z})$ corresponds to the pair of an abelian variety $\mathbf{C}^g/L$, where $L = \mathbf{Z}^g + \mathbf{Z}^g \cdot Z$ is the lattice in $\mathbf{C}^g$ generated by the unit vectors $\boldsymbol{e}_i$ and the i-th row vectors $\boldsymbol{z}_i$ of Z), and the polarization associated with the alternating bilinear form ψ on $L \times L$ such that

$$\psi(\boldsymbol{e}_i, \boldsymbol{e}_j) = \psi(\boldsymbol{z}_i, \boldsymbol{z}_j) = 0, \quad \psi(\boldsymbol{e}_i, \boldsymbol{z}_j) = \delta_{ij}.$$

By Torelli's theorem, by the correspondence:

proper smooth curves C

$\longmapsto$ their Jacobian varieties $\mathrm{Jac}(C)$ with principally polarization

induced from the theta divisor $\{P_1 + \cdots + P_{g-1} - (g-1)P_0 \mid P_i \in C\}$,

the (coarse) moduli of proper smooth curves is realized as a subvariety of $\mathcal{A}_g$. This fact gives rise to the **Schottky problem** which means to characterize Jacobian varieties among general abelian varieties, or to describe explicitly the subvariety of $\mathcal{A}_g$ consisting of Jacobian varieties.

3. Geometric invariant theory: For a proper smooth curve C over S of genus $g > 1$, the spaces $H^0(C_s, \Omega_{C_s}^{\otimes 3})$ $(s \in S)$ have dimension $5(g-1)$ by Riemann-Roch's theorem, and give an embedding $C \hookrightarrow \mathbb{P}_S^{5g-6}$. Then by the theory of Hilbert schemes, there exists a fine moduli scheme $\mathcal{H}_g$ over $\mathbf{Z}$ classifying tricanonically embedded curves $C \hookrightarrow \mathbb{P}_S^{5g-6}$, and hence the quotient stack

$$\mathcal{M}_g \stackrel{\text{def}}{=} [\mathcal{H}_g/PGL_{5g-5}]$$

is the fine moduli space of proper smooth curves of genus g. Since PGL_{5g-6} is smooth and the functor $S \mapsto \mathrm{Isom}_S(C, C')$ is represented by a finite and unramified scheme over S for curves C, C' over S, by an etale slice argument, $\mathcal{M}_g$ becomes an algebraic stack. Furthermore, by showing that each point on $\mathcal{H}_g$ is *stable* under the action of PGL_{5g-5}, it follows from **geometric invariant theory** [FKM] by Mumford that the geometric quotient $\mathcal{H}_g/PGL_{5g-5}$ exists and gives the coarse moduli scheme of proper smooth curves of genus g.

Dictionary for the moduli stack. In what follows,

$$\boldsymbol{\mathcal{M}_g \stackrel{\text{def}}{=} \text{the moduli stack over } \mathbf{Z} \text{ of proper smooth curves of genus } g > 1.}$$

Then

$$\mathcal{M}_g(\mathbf{C}) \ = \ \text{the quotient orbifold } [T_g/\Pi_g]\,,$$

and for schemes (more generally algebraic stacks) S,

$$\begin{aligned} & \mathcal{M}_g(S) = \text{the category of proper smooth curves over } S \text{ of genus } g \\ \Rightarrow \quad & \mathrm{Id}_{M_g} : \mathcal{M}_g \to \mathcal{M}_g \text{ gives the } \textbf{universal curve } \mathcal{C} \text{ over } \mathcal{M}_g. \end{aligned}$$

Furthermore,

$$\begin{aligned} & \text{an object } \alpha \text{ on (over) } \mathcal{M}_g \\ \Longleftrightarrow \quad & \text{a system } \{\alpha_S\} \text{ of objects on } S \text{ for proper smooth curves over } S \text{ of genus } g \\ & \text{such that } \{\alpha_S\} \text{ are functorial for } S. \end{aligned}$$

Dimension of the moduli.

- **Analytic method:** Since $\mathrm{Aut}(H_1) = PSL_2(\mathbf{R})$, by the theory of Fuchsian models, for Riemann surfaces $R = H_1/\pi_1(R)$, $R' = H_1/\pi_1(R')$ of genus $g > 1$,

$$\begin{aligned} & (R; \pi_1(R) \hookrightarrow PSL_2(\mathbf{R})) \cong \big(R; \pi_1(R') \hookrightarrow PSL_2(\mathbf{R})\big) \\ \Longleftrightarrow \quad & \pi_1(R) \text{ and } \pi_1(R') \text{ are conjugate in } PSL_2(\mathbf{R}). \end{aligned}$$

Therefore, under fixing a Riemann surface R_0 of genus g,

$$T_g \cong \left\{ \begin{array}{l} \text{conjugacy classes of injective homomorphisms} \\ \iota : \pi_1(R_0) \to PSL_2(\mathbf{R}) \text{ satisfying that} \\ H_1/\iota(\pi_1(R_0)) \text{ are Riemann surfaces of genus } g \end{array} \right\},$$

and the real dimension of the right hand side is

$$\begin{aligned} & \dim_{\mathbf{R}}(PSL_2(\mathbf{R}) \times (\sharp\{\text{generators of } \pi_1(R_0)\} - \sharp\{\text{relations in } \pi_1(R_0)\} - 1) \\ = \quad & 6g - 6. \end{aligned}$$

Furthermore, under the assumption that for Schottky uniformized Riemann surfaces R, R' of genus $g > 1$,

$$\begin{aligned} & (R = \Omega_\Gamma/\Gamma; \Gamma \hookrightarrow PGL_2(\mathbf{C})) \cong \big(R' = \Omega_{\Gamma'}/\Gamma'; \Gamma' \hookrightarrow PGL_2(\mathbf{C})\big) \\ \overset{\text{may be}}{\Longleftrightarrow} \quad & \Gamma \text{ and } \Gamma' \text{ are conjugate in } PGL_2(\mathbf{C}), \end{aligned}$$

by letting F_g be the free group of rank g, we have

$$\mathcal{M}_g(\mathbf{C}) \cong \left\{ \begin{array}{l} \text{conjugacy classes of injective homomorphisms} \\ \iota : F_g \to PGL_2(\mathbf{C}) \text{ satisfying that} \\ \iota(F_g) \text{ are Schottky groups} \end{array} \right\} \Big/ \mathrm{Aut}(F_g),$$

and the complex dimension of the right hand side is

$$\dim_{\mathbf{C}}(PGL_2(\mathbf{C}) \times (\sharp\{\text{generators of } F_g\} - 1) = 3g - 3.$$

- **Algebraic method (deformation theory [HrM]):** For a field k,

$$\begin{aligned} A_0 &\stackrel{\text{def}}{=} k[\varepsilon]/(\varepsilon^2), \\ C &: \text{ a proper smooth curve over } k \text{ of genus } g>1, \\ \{U_\alpha\} &: \text{ an affine open cover of } C, \end{aligned}$$

and let $\varphi_{\alpha\beta}$ be a first-order infinitesimal deformation of C, i.e., A_0-linear ring homomorphisms

$$\mathcal{O}_{U_\alpha\times\mathrm{Spec}(A_0)}|_{(U_\alpha\cap U_\beta)} \to \mathcal{O}_{U_\beta\times\mathrm{Spec}(A_0)}|_{(U_\alpha\cap U_\beta)}$$

satisfying that

$$\begin{cases} \varphi_{\alpha\gamma} = \varphi_{\beta\gamma}\circ\varphi_{\alpha\beta} \text{ on } U_\alpha\cap U_\beta\cap U_\gamma \text{ (: the cocycle condition)}, \\ \varphi_{\alpha\beta}|_{(U_\alpha\cap U_\beta)\times\mathrm{Spec}(k)} \text{ is the identity.} \end{cases}$$

Then the k-linear homomorphisms $D_{\alpha\beta} : \mathcal{O}_{(U_\alpha\cap U_\beta)} \to \mathcal{O}_{(U_\alpha\cap U_\beta)}$ given by $\varphi_{\alpha\beta}(f) = f + \varepsilon D_{\alpha\beta}(f)$ satisfies that

$$D_{\alpha\beta}(f\cdot g) = f\cdot D_{\alpha\beta}(g) + g\cdot D_{\alpha\beta}(f),\quad D_{\alpha\gamma}(f) = D_{\beta\gamma}(f)\cdot D_{\alpha\beta}(f),$$

and hence $\{D_{\alpha\beta}\}$ defines an element of the first cohomology $H^1(C,\mathcal{T}_C)$ of the tangent bundle $\mathcal{T}_C$ on C. Since $\dim_k(C)=1$, the obstruction space is $H^2(C,\mathcal{T}_C)=\{0\}$, and hence the tangent space of $\mathcal{M}_g\otimes_{\mathbf{Z}} k$ at the point $[C]$ corresponding to C is isomorphic to $H^1(C,\mathcal{T}_C)$. Therefore,

$$\begin{aligned} & \text{the dimension of the tangent space of } \mathcal{M}_g\otimes_{\mathbf{Z}} k \text{ at } [C] \\ = \;& \dim_k H^1(C,\mathcal{T}_C) \\ = \;& \dim_k H^0(C,\Omega_C^{\otimes 2}) \text{ (by Serre's duality)} \\ = \;& 3g-3 \text{ (by Riemann-Roch's theorem and that } \deg(\Omega_C) = 2g-2>0). \end{aligned}$$

Remark. For proper smooth curves C,

$$H^1(C,\mathcal{T}_C)\cong \mathrm{Ext}^1(\mathcal{O}_C,\mathcal{T}_C)\cong \mathrm{Ext}^1(\Omega_C,\mathcal{O}_C),$$

and the last group also classifies first-order infinitesimal deformations of stable curves.

3.2. Stable curves and their moduli

Stable curves. A **stable curve** of genus $g>1$ over a scheme S is a proper and flat morphism $C\to S$ whose geometric fibers are reduced and connected 1-dimensional schemes C_s such that

- C_s has only ordinary double points;

- $\mathrm{Aut}(C_s)$ is a finite group, i.e., if X is a smooth rational component of C_s, then X meets the other components of C_s at least 3 points;
- the dimension of $H^1(C_s, \mathcal{O}_{C_s})$ is equal to g.

For a stable curve C over S (may not be smooth), it is useful to consider the **dualizing sheaf** (or canonical invertible sheaf) $\omega_{C/S}$ on C which is defined as the following conditions:

- $\omega_{C/S}$ is functorial on S;
- if $S = \mathrm{Spec}(k)$ (k is an algebraically closed field), $f : C' \to C$ be the normalization (resolution) of C, $x_1, ..., x_n$, $y_1, ..., y_n$, are the points of C' such that $z_i = f(x_i) = f(y_i)$ $(1 \le i \le n)$ are the ordinary double points on C, then $\omega_{C/S}$ is the sheaf of 1-forms η on C' which are regular except for simple poles at x_i, y_i such that

$$\mathrm{Res}_{x_i}(\eta) + \mathrm{Res}_{y_i}(\eta) = 0.$$

Then it is shown by Rosenlicht and Hartshorne that $\omega_{C/S}$ is a line bundle on C, Riemann-Roch's theorem holds for the canonical divisor corresponding to ω_C, and

$$\dim H^1(C_s, \mathcal{O}_{C_s}) \ = \ \dim H^0(C_s, \omega_{C_s}).$$

Theorem 3.1. (Deligne and Mumford [DM]) *There exists the fine moduli space $\overline{\mathcal{M}}_g$ (called **Deligne-Mumford's compactification** of $\mathcal{M}_g$) as an algebraic stack over $\mathbf{Z}$ classifying stable curves of genus $g > 1$. $\overline{\mathcal{M}}_g$ is proper smooth over $\mathbf{Z}$, and contains $\mathcal{M}_g$ as its open dense substack.*

Sketch of proof. The construction of $\overline{\mathcal{M}}_g$ is similar to that of $\mathcal{M}_g$ by replacing Ω_C with dualizing sheaves ω_C. The properness of $\overline{\mathcal{M}}_g$ follows from the valuative criterion and the **stable reduction theorem**: Let R be a discrete valuation ring with quotient field K, and let C be a proper and smooth curve over K of genus $g > 1$. Then there exists a finite extension L of K and a stable curve $\mathcal{C}$ over the integral closure R_L of R in L such that $\mathcal{C} \otimes_{R_L} L \cong C \otimes_K L$.

Irreducibility of the moduli.

As an application of Theorem 3.1, Deligne and Mumford [DM] proved the irreducibility of any geometric fibers of $\overline{\mathcal{M}}_g$ by applying Enriques-Zariski's connectedness theorem to the proper and smooth stack $\overline{\mathcal{M}}_g$ over $\mathbf{Z}$ whose fiber over $\mathbf{C}$ is connected (by Teichmüller's theory). Therefore,

Any geometric fiber of $\mathcal{M}_g$ is irreducible.

This fact is essentially used in 4.3 to study automorphic forms on the moduli of curves.

3.3. Tate curve and Mumford curves

In order to study arithmetic geometry on $\mathcal{M}_g$, we want to

$$\begin{array}{rl} & \text{put local coordinates on } \mathcal{M}_g \\ \longleftrightarrow & \text{make a family of curves over the coordinate ring.} \end{array}$$

By the theory of the Tate curve and its higher genus version, we can put good coordinates near the boundary of $\mathcal{M}_g$ in terms of arithmetic geometry as follows.

Tate curve. Recall that an elliptic curve $\mathbf{C}/L$ is defined by the equation (see 2.1):

$$y^2 = 4x^3 - 60E_4(L)x - 140E_6(L).$$

Therefore, if

$$\begin{aligned} x &= (2\pi\sqrt{-1})^2 \left(x' + \frac{1}{12}\right), & y &= (2\pi\sqrt{-1})^3 \left(2y' + x'\right), \\ a_4 &= -\frac{15E_4(L)}{(2\pi\sqrt{-1})^4} + \frac{1}{48}, & a_6 &= -\frac{35E_6(L)}{(2\pi\sqrt{-1})^6} - \frac{5E_4(L)}{4(2\pi\sqrt{-1})^4} + \frac{1}{1728}, \end{aligned}$$

then the above equation is equivalent to

$$y'^2 + x'y' = x'^3 + a_4x' + a_6.$$

Furthermore, if $L = \mathbf{Z} + \mathbf{Z}\tau$ and $q = e^{2\pi\sqrt{-1}\tau}$, then by the calculation of the **Eisenstein series** (see Exercise 3.2 below):

$$\sum_{u \in L - \{0\}} \frac{1}{u^{2k}} = 2\zeta(2k) + \frac{2(2\pi\sqrt{-1})^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\ q^n \quad (k > 1),$$

where

$$\zeta(2k) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} : \text{ the \textbf{zeta values}, and } \sigma_{2k-1}(n) \stackrel{\text{def}}{=} \sum_{d|n} d^{2k-1},$$

we have

$$\begin{aligned} a_4(q) &= -5 \sum_{n=1}^{\infty} \sigma_3(n)\ q^n = -5q - 45q^2 + \cdots, \\ a_6(q) &= -\frac{1}{12} \sum_{n=1}^{\infty} (5\sigma_3(n) + 7\sigma_5(n))\ q^n = -q - 23q^2 + \cdots. \end{aligned}$$

Exercise 3.2. Prove that

$$\zeta(2k) = -\frac{(2\pi\sqrt{-1})^{2k}}{2(2k)!}B_{2k}$$

$$\left(B_n \text{ is the } n\text{-th } \textbf{Bernoulli numbers} \text{ given by } \frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\right)$$

$$\Rightarrow \quad \zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945},$$

and

$$\sum_{(m,n)\in\mathbf{Z}^2-\{(0,0)\}} \frac{1}{(m+n\tau)^{2k}} = 2\zeta(2k) + \frac{2(2\pi\sqrt{-1})^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)\ q^n \quad (k>1),$$

from the well-known formula:

$$\pi\cot(\pi a) = \frac{1}{a} + \sum_{m=1}^{\infty}\left(\frac{1}{a+m}+\frac{1}{a-m}\right) \quad \left(\Leftrightarrow\ \sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)\right)$$

by substituting x to $2\pi\sqrt{-1}a$, and differentiating the formula successively and substituting $n\tau$ to a respectively.

Exercise 3.3. Show that $a_4(q)$ and $a_6(q)$ belong to the ring

$$\mathbf{Z}[[q]] \stackrel{\text{def}}{=} \left\{ \sum_{n=0}^{\infty} c_n q^n \;\middle|\; c_n \in \mathbf{Z} \right\}$$

of formal power series of q with coefficients in $\mathbf{Z}$.

The **Tate curve** is the curve over $\mathbf{Z}[[q]]$ defined by

$$y^2 + xy = x^3 + a_4(q)x + a_6(q).$$

Then Tate proved the following:

Theorem 3.2. ([Si, T])

(1) *The Tate curve becomes an elliptic curve over the ring*

$$\mathbf{Z}((q)) \stackrel{\text{def}}{=} \mathbf{Z}[[q]]\,[1/q] = \left\{ \sum_{n>m}^{\infty} c_n q^n \;\middle|\; m\in\mathbf{Z},\ c_n\in\mathbf{Z} \right\}$$

of Laurent power series of q with coefficients in $\mathbf{Z}$.

(2) *Put*

$$X(u,q) = \sum_{n\in\mathbf{Z}} \frac{q^n u}{(1-q^n u)^2} - 2\sum_{n=1}^{\infty}\sigma_1(n)q^n,$$

$$Y(u,q) = \sum_{n\in\mathbf{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + \sum_{n=1}^{\infty}\sigma_1(n)q^n.$$

Then $z \mapsto \left(X(e^{2\pi\sqrt{-1}z}, e^{2\pi\sqrt{-1}\tau}), Y(e^{2\pi\sqrt{-1}z}, e^{2\pi\sqrt{-1}\tau})\right)$ *gives rise to an isomorphism between* $\mathbf{C}/L$ *and the elliptic curve* E_τ *over* $\mathbf{C}$ *obtained from the Tate curve by substituting* $q = e^{2\pi\sqrt{-1}\tau}$.

(3) *Let* K *be a complete valuation field with multiplicative valuation* $|\cdot|$, *and let* $q \in K^\times$ *satisfy that* $|q| < 1$. *Then by the substitution the variable* $q \mapsto q \in K^\times$, *the series* $a_4(q)$ *and* $a_6(q)$ *converge in* K, *and the Tate curve gives an elliptic curve* E_q *over* K. *Further, we have an isomorphism:*

$$\begin{array}{rcl} K^\times/\langle q\rangle & \xrightarrow{\sim} & E_q(K) \\ u \bmod \langle q\rangle & \longmapsto & \begin{cases} (X(u,q), Y(u,q)) & (u \notin \langle q\rangle), \\ 0 & (u \in \langle q\rangle). \end{cases} \end{array}$$

Proof. (1) The discriminant Δ of the Tate curve is given by

$$\begin{aligned} & -a_6(q) + a_4(q)^2 + 72a_4(q)a_6(q) - 64a_4(q)^3 - 432a_6(q)^2 \\ = \ & q - 24q^2 + \cdots : \text{ a formal power series with integral coefficients} \\ \overset{\text{in fact}}{=} \ & q\prod_{n=1}^{\infty}(1-q^n)^{24} : \text{ a cusp form of weight 12 for } SL_2(\mathbf{Z}). \end{aligned}$$

Therefore, the Tate curve is smooth over $\mathbf{Z}[[q]]\,[1/\Delta] = \mathbf{Z}((q))$.

(2) First, note that the following hold:

$$\begin{aligned} \frac{\wp_L(z)}{(2\pi\sqrt{-1})^2} &= \sum_{n\in\mathbf{Z}} \frac{q^n u}{(1-q^n u)^2} + \frac{1}{12} - 2s_1(q) \quad \left(s_1(q) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \sigma_1(n) q^n\right), \\ \frac{\wp_L'(z)}{(2\pi\sqrt{-1})^3} &= \sum_{n\in\mathbf{Z}} \frac{q^n u(1+q^n u)}{(1-q^n u)^3}, \end{aligned}$$

because the right hand sides are invariant under $u \mapsto qu$, hence invariant under $z \mapsto z+1, z+\tau$, and they have the same principal parts at $z = 0$ to the left hand sides. Therefore,

$$\begin{aligned} x' &= \frac{x}{(2\pi\sqrt{-1})^2} - \frac{1}{12} = \frac{\wp_L(z)}{(2\pi\sqrt{-1})^2} - \frac{1}{12} = X(u,q), \\ y' &= \frac{y}{2(2\pi\sqrt{-1})^3} - \frac{x}{2(2\pi\sqrt{-1})^2} + \frac{1}{24} \\ &= \frac{\wp_L'(z)}{2(2\pi\sqrt{-1})^3} - \frac{\wp_L(z)}{2(2\pi\sqrt{-1})^2} + \frac{1}{24} \\ &= \frac{1}{2}\sum_{n\in\mathbf{Z}} \frac{q^n u(1+q^n u)}{(1-q^n u)^3} - \frac{1}{2}\sum_{n\in\mathbf{Z}} \frac{q^n u}{(1-q^n u)^2} + s_1(q) \\ &= Y(u,q). \end{aligned}$$

As seen in 2.1, $z + L \mapsto (x = \wp_L(z), y = \wp'_L(z))$ is an isomorphism from $\mathbf{C}/L$ onto the elliptic curve $y^2 = 4x^3 - 60E_4(L)x - 140E_6(L)$, and hence

$$z + L \mapsto (x' = X(u, q), y' = Y(u, q))$$

gives an isomorphism $\mathbf{C}/L \xrightarrow{\sim} E_\tau$.

(3) By substituting the variable $q \mapsto q \in K^\times$ with $|q| < 1$, $\Delta = q - 24q^2 + \cdots$ satisfies that $|\Delta| = |q| \neq 0$, and hence E_q is an elliptic curve over K. By (2), $X(u, q)$ and $Y(u, q)$ satisfies the equation of the Tate curve:

$$Y(u, q)^2 + X(u, q)Y(u, q) = X(u, q)^3 + a_4(q)X(u, q) + a_6(q)$$

for all complex numbers u, q in a certain convergence domain, and hence this equation holds as formal power series in q with coefficients in $\mathbf{Q}(u)$. Therefore, by substituting the variable $q \mapsto q \in K^\times$ with $|q| < 1$, one can see that the map in (3) is well-defined, and is evidently injective. The addition law on the Tate curve is given by

$$\begin{array}{ll} & P_i = (x_i, y_i)\ (i = 1, 2, 3), \quad P_1 + P_2 = P_3 \\ \longrightarrow & \left\{ \begin{array}{lcl} (x_2 - x_1)^2 x_3 & = & (y_2 - y_1)^2 + (y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)^2(x_1 + x_2), \\ (x_2 - x_1)y_3 & = & (-(y_2 - y_1) + (x_2 - x_1))\, x_3 - (y_1 x_2 - y_2 x_1), \end{array} \right. \end{array}$$

if $x_1 \neq x_2$. Hence by (2), this holds if $x_i = X(u_i, q)$, $y_i = X(u_i, q)$, $(i = 1, 2, 3)$ with $u_1 u_2 = u_3$ for all complex numbers u_1, u_2, q in a certain convergence domain, and hence holds as formal power series in q with coefficients in $\mathbf{Q}(u_1, u_2)$. Therefore, by substituting the variable $q \mapsto q \in K^\times$ with $|q| < 1$, one can see that the map in (3) is a homomorphism. We omit the surjectivity of the map which is most hardest part of the proof. QED.

Remark. Similar argument to the proof of Theorem 3.2 (3) is used in [I1] to show that p-adic theta functions of Mumford curves give solutions to soliton equations.

Mumford curves. Mumford [Mu2] gave a higher genus version of the Tate curve over complete local domains as an analogy of Schottky uniformization theory, i.e., for a complete integrally closed noetherian local ring R with quotient field K, and a Schottky group $\Gamma \subset PGL_2(K)$ over K which is *flat* over R, he constructed a **Mumford curve** over $(R \subset)K$ which is a proper smooth curve C_Γ over K obtained as the general fiber of a stable curve over R uniformized by Γ such that its special fiber consists of (may be singular) projective lines and its singularities are all k-rational (k is the residue field of R). Furthermore, he showed that $\Gamma \mapsto C_\Gamma$ gives rise to the following bijection:

$$\left\{ \begin{array}{l} \text{Conjugacy classes of flat} \\ \text{Schottky groups over } (R \subset)K \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{Mumford curves over } (R \subset)K \end{array} \right\}$$

If K is a complete valuation field, then any Schottky group Γ over K is flat over its valuation ring, and it is shown in [Gv] that C_Γ is given as the quotient by Γ of its region of discontinuity in $K \cup \{\infty\}$ (important examples of rigid analytic geometry).

3.4. Arithmetic Schottky uniformization

Degenerate curves and dual graphs. A **degenerate curve** is a stable curve whose irreducible components are (may be singular) projective lines. For a degenerate curve, by the correspondence:

$$\begin{array}{rcl} \text{its irreducible components} & \longleftrightarrow & \text{vertices} \\ \text{its singular points} & \longleftrightarrow & \text{edges} \end{array}$$

(an irreducible component contains a singular point if and only if the corresponding vertex is contained in (or adjacent to) the corresponding edge), we have its **dual graph** which becomes a stable graph, i.e., a connected and finite graph whose vertices has at least 3 branches (**Figure**). For a degenerate curve C with dual graph Δ,

$$\begin{aligned} \text{the genus of } C &= \mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z}) \\ &= \text{the number of generators of the free group } \pi_1(\Delta). \end{aligned}$$

Since any triplet of distinct points on $\mathbb{P}^1$ is uniquely translated to $(0, 1, \infty)$ by the action of PGL_2, for a stable graph Δ, the moduli space of degenerate curves with dual graph Δ has dimension

$$\sum_{v:\ \text{vertices of}\ \Delta} (\deg(v) - 3)\,,$$

where $\deg(v)$ denotes the number of branches ($\neq$ edges) starting from v. In particular, a stable graph is **trivalent**, i.e., all the vertices have just 3 branches if and only if the corresponding curves are **maximally degenerate** which means that this moduli consists of only one point.

Exercise 3.4. For any stable graph Δ, prove that

$$\sum_{v:\ \text{vertices of}\ \Delta} (\deg(v) - 3) + \text{the number of edges of } \Delta \;=\; 3\left(\mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z}) - 1\right).$$

General degenerating process. (Ihara and Nakamura [IN]). For a stable graph Δ with orientation on each edge,

$$\begin{aligned} g &\stackrel{\text{def}}{=} \mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z}), \\ P_v &\stackrel{\text{def}}{=} \mathbb{P}^1(\mathbf{C}) \quad (v : \text{vertices of } \Delta). \end{aligned}$$

and for each oriented edge e ($v_{-e} \stackrel{e}{\to} v_e$) of Δ, let

$$\begin{array}{rcl} v_e & \stackrel{\text{def}}{=} & \text{the end point of } e, \\ v_{-e} & \stackrel{\text{def}}{=} & \text{the starting point of } e, \\ \gamma_e & : & \text{a hyperbolic element of } PGL_2(\mathbf{C}) \text{ which gives } \gamma_e : P_{v_{-e}} \stackrel{\sim}{\to} P_{v_e}, \\ t_e & \in & P_{v_e} : \text{the attractive fixed point of } \gamma_e, \\ t_{-e} & \in & P_{v_{-e}} : \text{the repulsive fixed point of } \gamma_e. \end{array}$$

Fix a vertex v_0 of Δ, and put

$$\Gamma \stackrel{\text{def}}{=} \left\{ \gamma_{e_1}^{i_1} \cdots \gamma_{e_n}^{i_n} \;\middle|\; e_k : \text{edges}, \, i_k \in \{\pm 1\} \text{ such that } e_n^{i_n} \cdots e_1^{i_1} \in \pi_1(\Delta; v_0) \right\}.$$

Then under the assumption that the multipliers s_e of all γ_e are sufficiently small,

- Γ is a Schottky group of rank g;
- If $\infty \in \Omega_\Gamma$, then $\sum_{\gamma \in \Gamma} |\gamma'(z)|$ converges uniformly on any compact subset of $\Omega_\Gamma - \bigcup_{\gamma \in \Gamma} \gamma(\infty)$;
- $R_\Gamma = \Omega_\Gamma / \Gamma$ is a Riemann surface of genus g obtained from holed Riemann spheres P_v (v : vertices of Δ) gluing by γ_e (e : edges of Δ);

and hence

$$\begin{array}{ll} & s_e \to 0 \ (e: \text{ edges of } \Delta) \\ \Rightarrow & R_\Gamma \to \text{the degenerate curve } C_0 = \left(\bigcup_v P_v \right) \Big/ \begin{array}{l} t_e = t_{-e} \\ (e: \text{ edges of } \Delta) \end{array} \quad \text{with dual graph } \Delta. \end{array}$$

Since $\mathbb{P}^1$ has only trivial deformation, R_Γ gives a universal deformation of C_0, and hence varying $t_{\pm e}$ as the **moduli parameters**, s_e as the **deformation parameters**, R_Γ make an open subset (of dimension $3g - 3$ by Exercise 3.4) of the moduli space of curves of genus g.

Arithmetic Schottky uniformization. An extension of this process in terms of arithmetic geometry (unifying complex geometry and formal geometry over $\mathbf{Z}$, hence rigid geometry) is the following **arithmetic Schottky uniformization theory** which also gives a higher genus version of the Tate curve:

Theorem 3.3. ([I3], (1)–(3) were already proved in [IN] for maximally degenerate case without singular components). *Let*

$$\begin{array}{rcl} A_0 & \stackrel{\text{def}}{=} & \textit{the coordinate ring of the moduli space (i.e., the ring of moduli parameters)} \\ & & \textit{over } \mathbf{Z} \textit{ of degenerate curves with dual graph } \Delta, \\ A_\Delta & \stackrel{\text{def}}{=} & A_0[[y_e \ (e: \textit{edges of } \Delta)]]. \end{array}$$

Then there exists a stable curve C_Δ (called the ***generalized Tate curve****) over A_Δ of genus $g \stackrel{\mathrm{def}}{=} \mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z})$ satisfying:*

(1) *C_Δ is a universal deformation of the universal degenerate curve with dual graph Δ.*

(2) *By substituting complex numbers $t_{\pm e}$ to the moduli parameters and $s_e \in \mathbf{C}^\times$ to y_e (e are edges of Δ), C_Δ becomes a Schottky uniformized Riemann surface if s_e are sufficiently small.*

(3) *C_Δ is smooth over $B_\Delta = A_\Delta\left[1/y_e \ (e : \text{edges of } \Delta)\right]$, and is Mumford uniformized by a Schottky group over B_Δ. Furthermore, for a complete integrally closed noetherian local ring R with quotient field K and a Mumford curve C over $(R \subset)K$ such that Δ is the dual graph of its degenerate reduction, there is a ring homomorphism $A_\Delta \to R$ gives rise to $C_\Delta \otimes_{A_\Delta} K \cong C$.*

(4) *Using Mumford's theory [Mu3] on degenerating abelian varieties, the generalized Jacobian of C_Δ can be expressed as*

$$\mathbf{G}_m^g / \langle (p_{ij})_{1 \leq i \leq g} \mid 1 \leq j \leq g \rangle; \quad \mathbf{G}_m \stackrel{\mathrm{def}}{=} \text{the multiplicative algebraic group},$$

where the multiplicative periods p_{ij} of C_Δ (called ***universal periods****) are given as computable elements of B_Δ.*

Generalized Tate curves

complex geometry ↙ ↘ rigid geometry

Schottky uniformized Riemann surfaces | Mumford curves

<u>*Sketch of proof.*</u>

- Step 1 of constructing C_Δ is to give a Schottky group Γ_Δ over B_Δ as in the above *general degenerating process,* and show that Γ_Δ is flat over A_Δ (note that this fact together with the result of [Mu2] cannot imply the existence of C_Δ since A_Δ is not local).

- Step 2 is, following argument in [Mu2], to show that the collection of sets consisting of 3 fixed points in $\mathbb{P}^1$ of $\Gamma - \{1\}$ gives rise to a tree which is the universal cover of Δ with covering group Δ, and to construct C_Δ as the quotient by Γ of the glued scheme of $\mathbb{P}^1_{A_\Delta}$ associated with this tree using Grothendieck's formal existence theorem.

- In order to give a power series expansion of p_{ij}, use the infinite product presentation by Schottky [S], Manin and Drinfeld [MD] of the multiplicative periods given in Theorem 2.2 (3).

Example 3.1. When Δ consists of one vertex and g loops, degenerate curves with dual graph Δ are obtained from $\mathbb{P}^1$ with $2g$ points $x_{\pm 1}, ..., x_{\pm g}$ by identifying $x_i = x_{-i}$ $(1 \leq i \leq g)$. Then

$$\begin{aligned} A_0 &= \mathbf{Z}\left[\frac{(x_i - x_j)(x_k - x_l)}{(x_i - x_l)(x_k - x_j)} \left(\begin{array}{ccc} i, j, k, l & \in & \{\pm 1, ..., \pm g\} \\ & : & \text{mutually different} \end{array} \right) \right], \\ A_\Delta &= A_0[[y_1, ..., y_g]], \end{aligned}$$

and C_Δ is uniformized by

$$\Gamma_\Delta \stackrel{\text{def}}{=} \left\langle \phi_i \stackrel{\text{def}}{=} \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y_i \end{pmatrix} \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \mathrm{mod}(\mathbf{G}_m) \;\middle|\; 1 \leq i \leq g \right\rangle.$$

Hence by Theorem 2.2 (3) (Exercise 2.5),

$$p_{ij} = \prod_{\phi \in \langle \phi_i \rangle \backslash \Gamma_\Delta / \langle \phi_j \rangle} \psi_{ij}(\phi),$$

where

$$\psi_{ij}(\phi) = \begin{cases} y_i & (\text{if } i = j \text{ and } \phi \in \langle \phi_i \rangle), \\ \dfrac{(x_i - \phi(x_j))(x_{-i} - \phi(x_{-j}))}{(x_i - \phi(x_{-j}))(x_{-i} - \phi(x_j))} & (\text{otherwise}). \end{cases}$$

Let I_Δ be the ideal of A_Δ generated by $y_1, ..., y_g$, and put $\phi_{-i} \stackrel{\text{def}}{=} \phi_i^{-1}$ $(1 \leq i \leq g)$. Then

$$\Phi_{ij} = \left\{ \phi = \phi_{\sigma(1)} \cdots \phi_{\sigma(n)} \;\middle|\; \begin{array}{l} \sigma(1) \neq \pm i, \ \sigma(n) \neq \pm j, \\ \sigma(k) \neq -\sigma(k+1) \ (1 \leq k \leq n-1) \end{array} \right\}$$

gives a set of complete representatives of $\langle \phi_i \rangle \backslash \Gamma_\Delta / \langle \phi_j \rangle$. For $\phi = \phi_{\sigma(1)} \cdots \phi_{\sigma(n)} \in \Phi_{ij}$, $\phi(x_{\pm j}) \in x_{\sigma(1)} + I_\Delta$, and

$$\begin{aligned} & \phi(x_j) - \phi(x_{-j}) \\ = \; & \frac{(x_{\sigma(1)} - x_{-\sigma(1)})^2 (\phi'(x_j) - \phi'(x_{-j})) y_{\sigma(1)}}{(\phi'(x_j) - x_{-\sigma(1)} - y_{\sigma(1)}(\phi'(x_j) - x_{\sigma(1)}))(\phi'(x_{-j}) - x_{-\sigma(1)} - y_{\sigma(1)}(\phi'(x_{-j}) - x_{\sigma(1)}))} \\ & (\phi' \stackrel{\text{def}}{=} \phi_{\sigma(2)} \cdots \phi_{\sigma(n)}) \\ = \; & \cdots \in I_\Delta^n. \end{aligned}$$

by inductive calculus, and hence

$$\frac{(x_i - \phi(x_j))(x_{-i} - \phi(x_{-j}))}{(x_i - \phi(x_{-j}))(x_{-i} - \phi(x_j))} = 1 + \frac{(x_i - x_{-i})(\phi(x_j) - \phi(x_{-j}))}{(x_i - \phi(x_{-j}))(x_{-i} - \phi(x_j))} \in 1 + I_\Delta^n.$$

Therefore,

$$p_{ij} = c_{ij} \left(1 + \sum_{|k| \neq i,j} \frac{(x_i - x_{-i})(x_j - x_{-j})(x_k - x_{-k})^2}{(x_i - x_k)(x_{-i} - x_k)(x_j - x_{-k})(x_{-j} - x_{-k})} y_{|k|} + \cdots \right),$$

where

$$c_{ij} \overset{\text{def}}{=} \begin{cases} y_i & (\text{if } i = j), \\ \dfrac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_i - x_{-j})(x_{-i} - x_j)} & (\text{if } i \neq j). \end{cases}$$

Remark. Denote by

T_g : the Teichmüller space of degree g,

S_g : the Schottky space of degree g
(the moduli space of Schottky groups with free g generators),

H_g : the Siegel upper half space of degree g.

Then

$$\begin{array}{ccll} T_g & \xrightarrow{p} & H_g & : \text{the period map (transcendental)} \\ \downarrow & & \downarrow \exp(2\pi\sqrt{-1}\cdot) & \\ S_g & \longrightarrow & H_g/\mathbf{Z}^{g(g+1)/2} & : \text{computable as power series} \\ \downarrow & & \downarrow & \\ \mathcal{M}_g(\mathbf{C}) & \xrightarrow{\tau} & H_g/Sp_{2g}(\mathbf{Z}) & : \text{the Torelli map (algebraic).} \end{array}$$

Exercise 3.5. Give a definition of the above period map $p : T_g \to H_g$.

Problem. When any vertex of Δ has just 3 branches (i.e., the corresponding degenerate curve is maximally degenerate), the moduli space of degenerate curves with dual graph Δ consists of one point, and hence $A_0 = \mathbf{Z}$. Then express integral coefficients of

$$p_{ij} \in A_\Delta = \mathbf{Z}[[y_e \ (e : \text{edges of } \Delta)]]$$

by using some arithmetic functions.

Problem. Give the equation of the Tate curve using Theorem 2.2 (1).

§4. Automorphic forms on the moduli space

4.1. Elliptic modular forms

The Eisenstein series of even degree $2k \geq 4$ (appeared in the Laurent coefficients of the $\wp$-function $\wp_{\mathbf{Z}+\mathbf{Z}\tau}(z)$):

$$E_{2k}(\tau) \overset{\text{def}}{=} \sum_{(m,n)\in\mathbf{Z}^2-\{(0,0)\}} \frac{1}{(m+n\tau)^{2k}} \overset{\textbf{Ex.3.1}}{=} 2\zeta(2k) + \frac{2(2\pi\sqrt{-1})^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)\ q^n$$

$$: \quad \text{the } \textbf{Fourier expansion } \left(q = e^{2\pi\sqrt{-1}\tau}\right)$$

is a holomorphic function of $\tau \in H$ which satisfies the following 2 conditions for $SL_2(\mathbf{Z})$:

- Automorphy condition of weight $2k$:
  $E_{2k}\left(\dfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k}E_{2k}(\tau)$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$;
- Cusp condition :
  $E_{2k}(\tau)$ is holomorphic at $q = 0$ $\big(\Leftrightarrow \tau =$ the unique cusp $\sqrt{-1}\cdot\infty$ of $SL_2(\mathbf{Z})\big)$.

(Elliptic) modular forms are holomorphic functions on H satisfying the automorphy and cusp conditions for a congruence subgroup of $SL_2(\mathbf{Z})$.

Fourier expansion and number theory.

- $\sigma_7(n) = \sigma_3(n) + 120\sum_{i=1}^{n-1}\sigma_3(i)\sigma_3(n-i)$ $\quad \left(\Leftarrow E_4(\tau)^2 = \dfrac{7}{9}E_8(\tau) \text{ in Exercise 2.1}\right)$.
- Jacobi's theorem : $\sharp\left\{(a_i)_{1\leq i\leq 4} \in \mathbf{Z}^4 \;\middle|\; \sum_{i=1}^{4} a_i^2 = n\right\} = 8\sum_{d|n, 4\nmid d} d$
  $\left(\Leftarrow \text{ the theta series } \left(\sum_{n\in\mathbf{Z}} q^{n^2}\right)^4 \text{ is expressed by Eisenstein series for } \Gamma(2)\right)$.
- Deligne-Serre's theorem [D1, DS]: For a normalized Hecke eigenform $f = \sum_n a(n)q^n$ of weight k and character ε for $\Gamma_0(N)$, there is a 2-dimensional Galois representation ρ_f such that $\mathrm{tr}(\rho_f(F_{\overline{p}})) = a(p)$ and $\det(\rho_f(F_{\overline{p}})) = \varepsilon(p)p^{k-1}$ for any Frobenius automorphism $F_{\overline{p}}$ for unramified primes p.
- Serre' example [Se]: Let L be the decomposition field of $x^3 - x - 1$ which is a Galois extension over $\mathbf{Q}$ with Galois group S_3 (: the symmetric group of degree 3) and contains $K = \mathbf{Q}(\sqrt{-23})$, and let

$$f(\tau) = \frac{1}{2}\left(\sum_{m,n\in\mathbf{Z}} q^{m^2+mn+6n^2} - \sum_{m,n\in\mathbf{Z}} q^{2m^2+mn+3n^2}\right) = \sum_{n=1}^{\infty} a(n)q^n.$$

Then $f(\tau)$ is a normalized Hecke eigenform of weight 1, and hence by Deligne-Serre's theorem, for any prime $p \neq 23$, $\mathrm{tr}(\rho_f(F_{\overline{p}})) = a(p)$, $\det(\rho_f(F_{\overline{p}})) = \left(\frac{-23}{p}\right) = \left(\frac{p}{23}\right)$ and $\sharp\langle \rho(F_{\overline{p}})\rangle$ is equal to the residue index $f_{L/\mathbf{Q}}(p)$ of p in $L/\mathbf{Q}$ (an example of nonabelian class field theory).

Exercise 4.1. Let the notation be as in the above Serre's example. Then prove that for $p \neq 23$, one of the following cases necessarily happens:

$$\begin{aligned} a(p) = 2,\ \left(\frac{p}{23}\right) = 1 &\iff f_{L/\mathbf{Q}}(p) = 1, \\ a(p) = 0,\ \left(\frac{p}{23}\right) = -1 &\iff f_{K/\mathbf{Q}}(p) = 2,\ f_{L/\mathbf{Q}}(p) = 2, \\ a(p) = -1,\ \left(\frac{p}{23}\right) = 1 &\iff f_{K/\mathbf{Q}}(p) = 1,\ f_{L/\mathbf{Q}}(p) = 3, \end{aligned}$$

and describe the decomposition of primes $2, 3, 5, 59$ in K and L respectively.

Rationality of modular forms. For $\tau \in H_1$,

$$\begin{aligned} E_\tau &\overset{\text{def}}{=} \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) \text{ define a family of elliptic curves over } H_1, \\ z_\tau &\overset{\text{def}}{=} \text{the natural coordinate of } \mathbf{C} \\ &\Rightarrow dz_\tau : \text{a canonical base of } H^0(E_\tau, \Omega_{E_\tau}), \end{aligned}$$

and

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \\ \Rightarrow\ & E_{\frac{a\tau+b}{c\tau+d}} \overset{\times(c\tau+d)}{\longrightarrow} \mathbf{C}/(\mathbf{Z}(c\tau+d) + \mathbf{Z}(a\tau+b)) = \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau) = E_\tau \\ \Rightarrow\ & dz_{\frac{a\tau+b}{c\tau+d}} = \frac{1}{c\tau+d} dz_\tau. \end{aligned}$$

If $f(\tau)$ is a modular form of weight k for $SL_2(\mathbf{Z})$, then

$$f\left(\frac{a\tau+b}{c\tau+d}\right)\left(d_{\frac{a\tau+b}{c\tau+d}}\right)^{\otimes k} = (c\tau+d)^k f(\tau)\left(\frac{1}{c\tau+d}\right)^k (dz_\tau)^{\otimes k} = f(\tau)(dz_\tau)^{\otimes k},$$

and hence $f(\tau)dz_\tau$ ($\tau \in H_1$) is $SL_2(\mathbf{Z})$-invariant, i.e., defines a holomorphic section of the line bundle on $[H_1/SL_2(\mathbf{Z})]$ whose fiber over $\tau \in H_1$ is given by $H^0(E_\tau, \Omega_{E_\tau})^{\otimes k}$.

Let $\mathcal{M}_1$ be the moduli stack of elliptic curves, $\pi : \mathcal{E} \to \mathcal{M}_1$ be the universal elliptic curve, and $\pi_*(\Omega_{\mathcal{E}/\mathcal{M}_1})$ denote a line bundle on $\mathcal{M}_1$ defined by the direct image of the sheaf $\Omega_{\mathcal{E}/\mathcal{M}_1}$ of relative 1-forms on $\mathcal{E}/\mathcal{M}_1$, i.e.,

$$\pi_*(\Omega_{\mathcal{E}/\mathcal{M}_1})(S) \overset{\text{def}}{=} H^0(E, \Omega_{E/S}),$$

for elliptic curves E over schemes S. Then an **integral modular form** f of weight k is defined as an element of

$$H^0\left(\mathcal{M}_1, \pi_*\left(\Omega_{\mathcal{E}}/\mathcal{M}_1\right)^{\otimes k}\right),$$

i.e., a global section of $\pi_*(\Omega_{\mathcal{E}/\mathcal{M}_1})^{\otimes k}$ on $\mathcal{M}_1$ which is, by the above dictionary on the moduli stack, a system of

$$\left\{\text{sections } f_S \text{ of } H^0\left(E, \Omega_{E/S}\right)^{\otimes k} \;\middle|\; E : \text{elliptic curves over } S\right\}$$

which are functorial for schemes S. Hence

$$\begin{aligned}
& E/S \;:\; \text{the Tate curve } y^2+xy=x^3+a_4(q)x+a_6(q) \text{ over } \mathbf{Z}((q)) \\
\Rightarrow\;& \frac{du}{u}=\frac{dX(u,q)}{X(u,q)+2Y(u,q)}=\frac{dx}{x+2y}\;:\; \text{a base of 1-forms on the Tate curve} \\
\Rightarrow\;& f \text{ is represented as } F(f)\left(\frac{dx}{x+2y}\right)^{\otimes k},
\end{aligned}$$

where $F(f)\in\mathbf{Z}((q))$ is called the **evaluation** of f on the Tate curve under the trivialization of $\pi_*\left(\Omega_{\mathcal{E}/\mathcal{M}_1}\right)$ on $\mathbf{Z}((q))$. By Theorem 3.2 (2),

$$\begin{aligned}
q=e^{2\pi\sqrt{-1}\tau} \quad &\Rightarrow\quad \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)\cong\mathbf{C}^\times/\langle q\rangle \\
&\Rightarrow\quad \frac{dx}{x+2y}=2\pi\sqrt{-1}\,\frac{d\wp_{\mathbf{Z}+\mathbf{Z}\tau}(z_\tau)}{\wp'_{\mathbf{Z}+\mathbf{Z}\tau}(z_\tau)}=2\pi\sqrt{-1}dz_\tau \\
&\Rightarrow\quad f(\tau)=(2\pi\sqrt{-1})^kF(f)(dz_\tau)^{\otimes k}.
\end{aligned}$$

Therefore, ignoring the factor $(2\pi\sqrt{-1})^k$,

the evaluation on the Tate curve = the classical Fourier expansion,

and hence

a modular form is integral $\Longleftrightarrow$ its Fourier coefficients are integral.

Exercise 4.2.

- Prove that $\dfrac{E_4(\tau)}{2\zeta(4)}$, $\dfrac{E_6(\tau)}{2\zeta(6)}$ and $\Delta(\tau)\overset{\text{def}}{=}\dfrac{1}{1728}\left(\left(\dfrac{E_4(\tau)}{2\zeta(4)}\right)^3-\left(\dfrac{E_6(\tau)}{2\zeta(6)}\right)^2\right)$ are integral (elliptic) modular forms for $SL_2(\mathbf{Z})$.
- Using that $\Delta(\tau)\neq 0$ ($\tau\in H$) and that modular forms for $SL_2(\mathbf{Z})$ of weight 0 are constant, prove that all integral modular forms for $SL_2(\mathbf{Z})$ are generated over $\mathbf{Z}$ by these 3 modular forms.

4.3. Siegel modular forms (SMFs)

Moduli of abelian varieties. Let g be a positive integer > 1. Then in a similar way to constructing moduli of curves given in 3.1, it is shown in [FKM] that there exists the fine moduli space $\mathcal{A}_g$ as an algebraic stack over $\mathbf{Z}$ classifying principally polarized abelian varieties of dimension g. By the correspondence:

$$Z = \begin{pmatrix} \boldsymbol{z}_1 \\ \vdots \\ \boldsymbol{z}_g \end{pmatrix} \in H_g : \text{ the Siegel upper half space of degree } g$$

$$\leftrightarrow \quad \left(\mathbf{C}^g / (\mathbf{Z}^g + \mathbf{Z}^g \cdot Z); \iota(\boldsymbol{e}_i) = \begin{cases} \boldsymbol{e}_i & (1 \leq i \leq g), \\ \boldsymbol{z}_{i-g} & (g+1 \leq i \leq 2g) \end{cases} \right),$$

H_g becomes the fine moduli space of principally polarized abelian varieties X of dimension g over $\mathbf{C}$ with symplectic isomorphism $\mathbf{Z}^{2g} \stackrel{\sim}{\to} H_1(X, \mathbf{Z})$. Hence the orbifold $\mathcal{A}_g(\mathbf{C})$ is given by the quotient stack of H_g by the integral symplectic group $Sp_{2g}(\mathbf{Z})$ of degree g :

$$\mathcal{A}_g(\mathbf{C}) \;=\; [H_g / Sp_{2g}(\mathbf{Z})] \,.$$

Definition of SMFs. Let λ be the **Hodge line bundle** on $\mathcal{A}_g$ which is defined by

$$\lambda \stackrel{\text{def}}{=} \bigwedge^{g} \rho_* \left(\Omega_{\mathcal{X}/\mathcal{A}_g} \right) \quad (\rho : \mathcal{X} \to \mathcal{A}_g \text{ denotes the universal abelian scheme})$$

$$\Rightarrow \quad \lambda(S) = \bigwedge^{g} H^0 \left(X, \Omega_{X/S} \right) \quad \text{for abelian schemes } X/S \text{ of relative dimension } g.$$

Then for $h \in \mathbf{Z}$ and a $\mathbf{Z}$-module M, we call elements of

$$S_{g,h}(M) \stackrel{\text{def}}{=} H^0 \left(\mathcal{A}_g, \lambda^{\otimes h} \otimes_{\mathbf{Z}} M \right)$$

Siegel modular forms of degree g and weight h with coefficients in M.

For the natural coordinate $z_1, ..., z_g$ on the complex abelian varieties $X_Z = \mathbf{C}^g / (\mathbf{Z}^g + \mathbf{Z}^g \cdot Z)$, $dz_1, ..., dz_g$ give a base of $H^0(X_Z, \Omega_{X_Z})$, and hence as in the elliptic case,

$$\varphi = (2\pi\sqrt{-1})^{gh} \cdot f \cdot (dz_1 \wedge \cdots \wedge dz_g)^{\otimes h} \in S_{g,h}(\mathbf{C}) = H^0 \left([H_g / Sp_{2g}(\mathbf{Z})] \,, \lambda^{\otimes h} \right)$$

$$\stackrel{(*)}{\Rightarrow} \quad \begin{cases} f = f(Z) \text{ is a holomorphic function of } Z \in H_g \text{ such that} \\ f(G(Z)) = \det(CZ+D)^h \cdot f(Z) \text{ for any } G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbf{Z}) \end{cases}$$

which is known as the usual definition of **analytic Siegel modular forms**. In particular, $f(Z)$ is invariant under the transformation

$$Z \longmapsto Z + B$$

by integral symmetric matrices B of degree g, and hence it can be expanded as a power series of $\exp\left(2\pi\sqrt{-1}z_{ij}\right)$ $(Z=(z_{ij})_{i,j}\in H_g)$ which is called the **(classical) Fourier expansion** of f.

Exercise 4.3. Prove the above $\overset{(*)}{\Rightarrow}$.

It is shown by Satake that $\mathcal{A}_{g/\mathbf{C}}=\mathcal{A}_g\otimes_{\mathbf{Z}}\mathbf{C}$ has the **Satake compactification**:

$$\mathcal{A}^*_{g/\mathbf{C}}=\coprod_{i=0}^{g}\mathcal{A}_{i/\mathbf{C}},$$

obtained as the Zariski closure of a projective embedding using Siegel modular forms of sufficiently large weight. Then the codimension of $\mathcal{A}^*_{g/\mathbf{C}}-\mathcal{A}_{g/\mathbf{C}}$ in $\mathcal{A}^*_{g/\mathbf{C}}$ is

$$\frac{g(g+1)}{2}-\frac{(g-1)g}{2}=g>1,$$

and hence ignoring $(2\pi\sqrt{-1})^{gh}\left(dz_1\wedge\cdots\wedge dz_g\right)^{\otimes h}$,

$$\begin{aligned}
&\varphi \text{ is an analytic Siegel modular form}\\
\Rightarrow\ &\varphi \text{ is an analytic section on } \mathcal{A}^*_{g/\mathbf{C}} \quad \text{(by Hartogs' theorem)}\\
\Rightarrow\ &\varphi \text{ is an algebraic section on } \mathcal{A}^*_{g/\mathbf{C}} \quad \text{(by GAGA's principle of Serre)}\\
\Rightarrow\ &\varphi \text{ is an algebraic section on } \mathcal{A}_{g/\mathbf{C}}\\
\Rightarrow\ &\varphi\in S_{g,h}(\mathbf{C}).
\end{aligned}$$

Therefore, the above $\overset{(*)}{\Rightarrow}$ is in fact an equivalence $\overset{(*)}{\Longleftrightarrow}$, and $S_{g,h}(\mathbf{C})$ is finite dimensional over $\mathbf{C}$ by the compactness of $\mathcal{A}^*_{g/\mathbf{C}}$.

Fourier expansion of SMFs. By Mumford's theory [Mu3] on degenerating abelian varieties, there exists a semiabelian scheme expressed as

$$\mathbf{G}_m^g/\left\langle (q_{ij})_{1\le i\le g}\mid 1\le j\le g\right\rangle$$

over the ring

$$\mathbf{Z}\left[q_{ij}^{\pm1}\ (i\neq j)\right][[q_{11},...,q_{gg}]],$$

where q_{ij} $(1\le i,j\le g)$ are variables with symmetry $q_{ij}=q_{ji}$. This semiabelian scheme gives a family of complex abelian varieties

$$\mathbf{C}^g/\left(\mathbf{Z}+\mathbf{Z}\cdot Z\right)\ \cong\ \left(\mathbf{C}^\times\right)^g\Big/\!\!\Big/\left\langle\left(\exp(2\pi\sqrt{-1}z_{ij})\right)_{1\le i\le g}\mid 1\le j\le g\right\rangle$$

when $q_{ij}=\exp(2\pi\sqrt{-1}z_{ij})$ for $Z=(z_{ij})_{i,j}\in H_g$. Then the natural coordinates $u_1,...,u_g$ on $\mathbf{G}_m^g$ give a base $du_1/u_1,...,du_g/u_g$ of 1-forms on this semiabelian scheme, and hence the evaluation of any $\varphi \in S_{g,h}(M)$ gives

$$\begin{aligned}\varphi &= F(\varphi)\cdot((du_1/u_1)\wedge\cdots\wedge(du_g/u_g))^{\otimes h} \\ &= (2\pi\sqrt{-1})^{gh}\cdot F(\varphi)\cdot(dz_1\wedge\cdots\wedge dz_g)^{\otimes h} \quad (\text{if } M=\mathbf{C} \text{ and } u_i=\exp(2\pi\sqrt{-1}z_i)).\end{aligned}$$

Therefore, we have a linear map:

$$F: S_{g,h}(M) \longrightarrow \mathbf{Z}\left[q_{ij}^{\pm 1}\ (i\neq j)\right][[q_{11},...,q_{gg}]]\otimes_{\mathbf{Z}} M,$$

which we call the **arithmetic Fourier expansion.**

Theorem 4.1. (Chai and Faltings [FC])

(1) *(Arithmetic Fourier expansion) F is functorial for M, and if $M=\mathbf{C}$, then $F(\varphi)$ is the classical Fourier expansion by $q_{ij}=\exp\left(2\pi\sqrt{-1}z_{ij}\right)$ for $(z_{ij})_{i,j}\in H_g$. Furthermore, F is injective, and for a submodule N of M and $\varphi\in S_{g,h}(M)$,*

$$\varphi\in S_{g,h}(N) \iff F(\varphi)\in\mathbf{Z}\left[q_{ij}^{\pm 1}\right][[q_{ii}]]\otimes_{\mathbf{Z}} M.$$

(2) *(Finiteness) $S_{g,h}(\mathbf{Z})$ is a free $\mathbf{Z}$-module of finite rank such that $S_{g,h}(\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{C} = S_{g,h}(\mathbf{C})$ and that $S_{g,0}(\mathbf{Z})=\mathbf{Z}$, $S_{g,h}(\mathbf{Z})=\{0\}$ if $n<0$. Furthermore, the ring of integral Siegel modular forms of degree g over $\mathbf{Z}$:*

$$S_g^*(\mathbf{Z}) \overset{\text{def}}{=} \bigoplus_{h\geq 0} S_{g,h}(\mathbf{Z})$$

is a normal ring finitely generated over $\mathbf{Z}$.

Sketch of Proof. (1) The functoriality for M and the compatibility with the classical Fourier expansion is clear from the above. Since $\mathcal{A}_g$ is smooth over $\mathbf{Z}$, we have the following left exactness of $S_{g,h}(M)$ for M :

$$\begin{aligned} & 0\to N\to M\to (M/N)\to 0 \\ \Rightarrow\ & 0\to\lambda^{\otimes h}\otimes_{\mathbf{Z}} N\to\lambda^{\otimes h}\otimes_{\mathbf{Z}} M\to\lambda^{\otimes h}\otimes_{\mathbf{Z}}(M/N)\to 0 \\ \Rightarrow\ & 0\to S_{g,h}(N)\to S_{g,h}(M)\to S_{g,h}(M/N).\end{aligned}$$

We prove the injectivity of F. Since any $\mathbf{Z}$-module M is the direct limit of finitely generated $\mathbf{Z}$-modules, and cohomology and tensor product commute with direct limit, we may assume that M is a finitely generated $\mathbf{Z}$-module, hence by the left exactness for M, we may put $M=\mathbf{Z}$ or $=\mathbf{Z}/p\mathbf{Z}$ (p : a prime number). Therefore, the injectivity follows from that $\mathcal{A}_g\otimes M$ is smooth over the ring M with geometrically irreducible fibers which is proved in [FC]. Hence the remains of (1) follows from this injectivity and the left exactness of $S_{g,h}$.

(2) is derived by the following result in [FC]: there exists an algebraic stack $\overline{\mathcal{A}}_g$ which is proper smooth over $\mathbf{Z}$ and contains $\mathcal{A}_g$ as its open dense substack, and any integral Siegel modular form of weight k can be extended to a section on $\overline{\mathcal{A}}_g$ of an extension $\overline{\lambda}^{\otimes k}$ of $\lambda^{\otimes k}$ (called **Koecher's principle**).

The finiteness of $\mathrm{rank}_{\mathbf{Z}} S_{g,h}(\mathbf{Z})$ follows from these results immediately. Further, there is $m \in \mathbf{N}$ such that $\overline{\lambda}^{\otimes m}$ defines a projective morphism $\overline{\mathcal{A}}_g \to \mathbb{P}^n_{\mathbf{Z}}$ which can be, by the theory of Stein factorization, decomposed as $\overline{\mathcal{A}}_g \to \mathcal{A}_g^* \to \mathbb{P}^n_{\mathbf{Z}}$ such that $\overline{\mathcal{A}}_g \to \mathcal{A}_g^*$ has connected geometric fibers and $\mathcal{A}_g^* \to \mathbb{P}^n_{\mathbf{Z}}$ is finite. Therefore, replacing m by a multiple $\overline{\lambda}^{\otimes m}$ defines a immersion of $\mathcal{A}_g^*$, and hence $\bigoplus_{k \geq 0} H^0(\overline{\mathcal{A}}_g, \overline{\lambda}^{\otimes mk})$ and $S_g^*(\mathbf{Z})$ are normal rings finitely generated over $\mathbf{Z}$. QED.

Ring of SMFs of degree 2 and 3. (Igusa [Ig1,3], Tsuyumine [Ty1]) For $g > 1$ and $h > g+1$, the **Eisenstein series** of degree $g > 1$ and weight h is a function of $Z \in H_g$ defined by

$$E_{g,h}(Z) \stackrel{\mathrm{def}}{=} \sum_{G \in \Gamma_\infty \backslash Sp_{2g}(\mathbf{Z})} \det{(CZ+D)^{-h}}; \quad G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$\Gamma_\infty \stackrel{\mathrm{def}}{=} \left\{ \begin{pmatrix} U & B \\ 0 & {}^tU^{-1} \end{pmatrix} \in Sp_{2g}(\mathbf{Z}) \right\}.$$

Then $E_{g,h}$ becomes a Siegel modular form with Fourier coefficients in $\mathbf{Q}$, and hence an element of $S_{g,h}(\mathbf{Q})$. Igusa [Ig1] proved that

$$S_2^*(\mathbf{C}) \ = \ \mathbf{C}\left[E_4, E_6, \Delta_{10}, \Delta_{12}\right] \bigoplus \Delta_{35} \cdot \mathbf{C}\left[E_4, E_6, \Delta_{10}, \Delta_{12}\right],$$

where $E_h = E_{2,h}$, $\Delta_{10} = E_4E_6 - E_{10}$, $\Delta_{12} = 441E_4^3 + 250E_6^2 - 691E_{12}$ and $\Delta_{35} \in S_{2,35}(\mathbf{C})$ is given by Ibukiyama as

$$\Delta_{35}\begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} = \begin{vmatrix} 4E_4 & 6E_6 & 10\Delta_{10} & 12\Delta_{12} \\ \dfrac{\partial E_4}{\partial z_{11}} & \dfrac{\partial E_6}{\partial z_{11}} & \dfrac{\partial \Delta_{10}}{\partial z_{11}} & \dfrac{\partial \Delta_{12}}{\partial z_{11}} \\ \dfrac{\partial E_4}{\partial z_{12}} & \dfrac{\partial E_6}{\partial z_{12}} & \dfrac{\partial \Delta_{10}}{\partial z_{12}} & \dfrac{\partial \Delta_{12}}{\partial z_{12}} \\ \dfrac{\partial E_4}{\partial z_{22}} & \dfrac{\partial E_6}{\partial z_{22}} & \dfrac{\partial \Delta_{10}}{\partial z_{22}} & \dfrac{\partial \Delta_{12}}{\partial z_{22}} \end{vmatrix}.$$

4.3. Teichmüller modular forms (TMFs)

Analytic	:	automorphic functions on the Teichmüller space
	=	automorphic forms on the moduli space of Riemann surfaces,
Algebraic	:	global sections of line bundles on the moduli of curves.

This naming is an analogy of

Siegel modular forms (SMFs)
= automorphic functions on the Siegel upper half space
= global sections of line bundles
on the moduli of principally polarized abelian varieties.

Definition of TMFs. Let $\pi : \mathcal{C} \to \mathcal{M}_g$ be the universal curve over the moduli stack of proper smooth curves of genus $g > 1$, and let $\lambda \stackrel{\text{def}}{=} \bigwedge^g \pi_* \left(\Omega_{\mathcal{C}/\mathcal{M}_g}\right)$ be the **Hodge line bundle**. Then for a $\mathbf{Z}$-module M, we call elements of

$$T_{g,h}(M) \stackrel{\text{def}}{=} H^0(\mathcal{M}_g, \lambda^{\otimes h} \otimes_{\mathbf{Z}} M)$$

Teichmüller modular forms of degree g and weight h with coefficients in M. By the pullback of the Torelli map $\tau : \mathcal{M}_g \to \mathcal{A}_g$ sending curves to their Jacobian varieties with canonical polarization, we have a linear map

$$\tau^* : S_{g,h}(M) \longrightarrow T_{g,h}(M)$$

for $\mathbf{Z}$-modules M. If $g = 2, 3$, then the image of the Torelli map is Zariski dense, and hence τ^* is injective.

If $n \geq 3$, then

$$\begin{aligned} \mathcal{M}_{g,n/\mathbf{C}} &\stackrel{\text{def}}{=} \text{the moduli space of proper smooth curves over } \mathbf{C} \\ &\quad\ \text{of genus } g \text{ with symplectic level } n \text{ structure,} \\ \mathcal{A}_{g,n/\mathbf{C}} &\stackrel{\text{def}}{=} \text{the moduli space of principally polarized abelian varieties over } \mathbf{C} \\ &\quad\ \text{of dimension } g \text{ with symplectic level } n \text{ structure} \end{aligned}$$

are given as fine moduli schemes over $\mathbf{C}$. Let $\mathcal{M}^*_{g,n/\mathbf{C}}$ be the **Satake-type** compactification, i.e., normalization of the Zariski closure of

$$(\iota \circ \tau)(\mathcal{M}_{g,n/\mathbf{C}}) \subset \mathcal{A}^*_{g,n/\mathbf{C}},$$

where $\tau : \mathcal{M}_{g,n/\mathbf{C}} \to \mathcal{A}_{g,n/\mathbf{C}}$ denote the Torelli map, and $\iota : \mathcal{A}_{g,n/\mathbf{C}} \to \mathcal{A}^*_{g,n/\mathbf{C}}$ denote the natural inclusion to the Satake compactification. Then each point of $\mathcal{M}^*_{g,n/\mathbf{C}} - \mathcal{M}_{g,n/\mathbf{C}}$ corresponds to the product $J_1 \times \cdots \times J_m$ of Jacobian varieties over $\mathbf{C}$ with canonical polarization and symplectic level n structure such that $\sum_{i=1}^m \dim(J_i) \leq g$ and that $(m, g) \neq (1, \dim(J_1))$. Therefore, if $g \geq 3$, then $\mathcal{M}^*_{g,n/\mathbf{C}} - \mathcal{M}_{g,n/\mathbf{C}}$ has codimension 2 in $\mathcal{M}^*_{g,n/\mathbf{C}}$, and hence by applying Hartogs' theorem to $\mathcal{M}_{g,n/\mathbf{C}} \subset \mathcal{M}^*_{g,n/\mathbf{C}}$ and GAGA's principle to $\mathcal{M}^*_{g,n/\mathbf{C}}$, one can see that analytic TMFs become algebraic TMFs, i.e.,

$$T_{g,h}(\mathbf{C}) \cong \left\{ \begin{array}{l} \text{holomorphic functions on the Teichmüller space } T_g \\ \text{with automorphy condition of weight } h \\ \text{for the action of the Teichmüller modular group } \Pi_g \end{array} \right\},$$

and this space is finite dimensional over $\mathbf{C}$.

Exercise 4.4. Give a precise definition of analytic Teichmüller modular forms.

Expansion of TMFs. Let C_Δ be the generalized Tate curve given in Theorem 3.3 which is smooth over the ring B_Δ. Then as in the elliptic and Siegel modular case, the evaluation on C_Δ (= the expansion by the corresponding local coordinates on $\mathcal{M}_g$) gives rise to a homomorphism

$$\kappa_\Delta : T_{g,h}(M) \longrightarrow B_\Delta \otimes_{\mathbf{Z}} M.$$

Theorem 4.2. ([I3]). *Fix $g > 1$ and $h \in \mathbf{Z}$.*

(1) *κ_Δ is injective, and for a Teichmüller modular form $f \in T_{g,h}(M)$ and a submodule N of M,*

$$f \in T_{g,h}(N) \iff \kappa_\Delta(f) \in B_\Delta \otimes_{\mathbf{Z}} N.$$

(2) *For a Siegel modular form $\varphi \in S_{g,h}(M)$,*

$$\kappa_\Delta(\tau^*(\varphi)) \;=\; F(\varphi)|_{q_{ij}=p_{ij}},$$

where p_{ij} are the multiplicative periods of C_Δ given in Theorem 3.3 (4).

Proof. (1) follows from the fact that C_Δ corresponds to the generic point on $\mathcal{M}_g$, and the argument in the proof of Theorem 4.1 (1) replacing $\mathcal{A}_g$ by $\mathcal{M}_g$ which is proper and smooth over $\mathbf{Z}$ with geometrically irreducible fibers (see 3.2). (2) follows from Theorem 3.3 (4). QED.

Schottky problem. As an application of Theorem 4.2, we can give a solution to the Schottky problem, i.e. characterizing Siegel modular forms vanishing on the Jacobian locus, is given as follows:

$$\tau^*(\varphi) = 0 \iff F(\varphi)|_{q_{ij}=p_{ij}} = 0.$$

p_{ij} are computable, hence κ_Δ are computable

Using the universal periods p_{ij} in Example 3.1, the above implies the following result of Brinkmann and Gerritzen [BG, G]: For the Fourier expansion

$$F(\varphi) = \sum_{T=(t_{ij})} a_T \prod_{1\le i<j\le g} {q_{ij}}^{2t_{ij}} \prod_{1\le i\le g} {q_{ii}}^{t_{ii}}$$

of a Siegel modular form φ vanishing on the Jacobian locus,

$$\text{integers } s_1, ..., s_g \geq 0 \text{ satisfy } \sum_{i=1}^{g} s_i = \min\{\mathrm{T}(T) \mid a_T \neq 0\}$$

$$\Rightarrow \sum_{t_{ii}=s_i} a_T \prod_{i<j} \left(\frac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_i - x_{-j})(x_{-i} - x_j)} \right)^{2t_{ij}} = 0 \text{ in } A_0 \text{ (: given in Example 3.1).}$$

Schottky' J. For $n \equiv 0 \bmod(4)$, put

$$L_{2n} \overset{\text{def}}{=} \left\{ (x_1, ..., x_{2n}) \in \mathbf{R}^{2n} \,\middle|\, 2x_i,\ x_i - x_j,\ \frac{1}{2}\sum_i x_i \in \mathbf{Z} \right\}$$

: a lattice in $\mathbf{R}^{2n}$ with standard inner product,

$$\varphi_n(Z) \overset{\text{def}}{=} \sum_{(\lambda_1,...,\lambda_4)\in L_{2n}^4} \exp\left(\pi\sqrt{-1} \sum_{i,j=1}^{4} \langle \lambda_i, \lambda_j \rangle z_{ij} \right) \quad (Z = (z_{ij})_{i,j} \in H_4)$$

: a Siegel modular form of degree 4 and weight n,

$$J(Z) \overset{\text{def}}{=} \frac{2^2}{3^2 \cdot 5 \cdot 7}(\varphi_4(Z)^2 - \varphi_8(Z)) : \textbf{ Schottky's } \boldsymbol{J}$$

: an integral Siegel modular form of degree 4 and weight 8.

Then Schottky and Igusa proved that the Zariski closure of the Jacobian locus in $\mathcal{A}_4 \otimes_{\mathbf{Z}} \mathbf{C}$ is defined by $J = 0$.

Brinkmann and Gerritzen [BG, G] checked the above Brinkmann and Gerritzen's criterion for Schottky's J, i.e., computed the lowest term of J and showed that this is given by up to contant

$$F \frac{q_{11}q_{22}q_{33}q_{44}}{\prod_{1 \leq i < j \leq 4} q_{ij}},$$

where F is a generator of the ideal of $\mathbf{C}\,[q_{ij}\ (1 \leq i < j \leq 4)]$ which is the kernel of the ring homomorphism given by

$$q_{ij} \mapsto \frac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_i - x_{-j})(x_{-i} - x_j)} \in A_0.$$

Problem. Let J' be a primitive modular form obtained from J by dividing the GCM (greatest common divisor) of its Fourier coefficients. Then for each prime p,

the closed subset of $\mathcal{A}_4 \otimes_{\mathbf{Z}} \mathbf{F}_p$ defined by $J' \bmod(p) = 0$

$\overset{?}{=}$ the Zariski closure of $\tau(\mathcal{M}_4 \otimes_{\mathbf{Z}} \mathbf{F}_p)$ in $\mathcal{A}_4 \otimes_{\mathbf{Z}} \mathbf{F}_p$.

Hyperelliptic Schottky problem. ([I4]) Let p_{ij} be the universal periods given in Example 3.1. Then

$$p'_{ij} \overset{\text{def}}{=} p_{ij}|_{x_{-k}=-x_k} \quad (1 \leq k \leq g)$$

become the multiplicative periods of the hyperelliptic curve $C_{\rm hyp}$ over

$$\mathbf{Z}\left[\frac{1}{2x_i}, \ \frac{1}{x_i \pm x_j}(i \neq j)\right][[y_1, ..., y_g]]$$

uniformaized by the Schottky group:

$$\left\langle \left(\begin{array}{cc} x_k & -x_k \\ 1 & 1 \end{array} \right) \left(\begin{array}{cc} 1 & 0 \\ 0 & y_k \end{array} \right) \left(\begin{array}{cc} x_k & -x_k \\ 1 & 1 \end{array} \right)^{-1} \ \middle| \ k = 1, ..., g \right\rangle.$$

Since $C_{\rm hyp}$ is generic in the moduli space of hyperelliptic curves, for any Siegel modular form φ over a field of characteristic $\neq 2$,

$$\varphi \text{ vanishes on the locus of hyperelliptic Jacobians} \iff F(\varphi)|_{q_{ij}=p'_{ij}} = 0.$$

Problem. Give an explicit lower bound of $n(g) \in \mathbf{N}$ satisfying that

$$\varphi \text{ vanishes on the locus of hyperelliptic Jacobians} \iff F(\varphi)|_{q_{ij}=p'_{ij}} \in I^{n(g)},$$

where I is the ideal generated by $y_1, ..., y_g$.

4.4. TMFs and geometry of the moduli

Theta constants and ring structure.

For $g \geq 2$, let

$$\theta_g(Z) \ \stackrel{\rm def}{=} \prod_{\substack{\boldsymbol{a}, \boldsymbol{b} \in \{0, 1/2\}^g \\ 4\boldsymbol{a}^t\boldsymbol{b} \,:\, \text{even}}} \sum_{\boldsymbol{n} \in \mathbf{Z}^g} \exp\left(2\pi\sqrt{-1}\left[\frac{1}{2}(\boldsymbol{n}+\boldsymbol{a})Z^t(\boldsymbol{n}+\boldsymbol{a}) + (\boldsymbol{n}+\boldsymbol{a})^t\boldsymbol{b}\right]\right)$$

be the product of even **theta constants** of degree g. If $g \geq 3$, then θ_g is an integral Siegel modular form of degree g and weight $2^{g-2}(2^g+1)$.

Theorem 4.3. ([I2, 3]). *For $g \geq 3$,*

(1) *$T_{g,h}(\mathbf{Z})$ is a free $\mathbf{Z}$-module of finite rank satisfying that $T_{g,h}(\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = T_{g,h}(\mathbf{C})$, and that $T_{g,0}(\mathbf{Z}) = \mathbf{Z}$, $T_{g,h}(\mathbf{Z}) = \{0\}$ if $h < 0$. Furthermore, the ring of integral Teichmüller modular forms of degree g :*

$$T_g^*(\mathbf{Z}) \stackrel{\rm def}{=} \bigoplus_{h \geq 0} T_{g,h}(\mathbf{Z})$$

becomes a normal ring which is finitely generated over $\mathbf{Z}$.

(2) *For the product θ_g of even theta constants of degree g,*

$$N_g \ \stackrel{\rm def}{=} \ \begin{cases} -2^{28} & (g=3), \\ 2^{2^{g-1}(2^g-1)} & (g \geq 4). \end{cases}$$

Then $\sqrt{\tau^*(\theta_g)/N_g}$ *is a* ***primitive*** *element of* $T_{g,2^{g-3}(2^g+1)}(\mathbf{Z})$*, i.e., not congruent to* 0 *modulo any prime.*

(3) $T_3^*(\mathbf{Z})$ *is generated by Siegel modular forms over* $\mathbf{Z}$ *and by* $\sqrt{\tau^*(\theta_3)/N_3}$ *which is of weight* 9*, hence is not a Siegel modular form.*

Proof. (1) follows from the argument in the proof of Theorem 4.1 (2) replacing

$$\left(\mathcal{A}_g,\ \overline{\mathcal{A}}_g,\ \overline{\lambda}\right) \quad \text{by} \quad \left(\mathcal{M}_g,\ \overline{\mathcal{M}}_g,\ \bigwedge^g \pi_*(\omega_{\mathcal{C}/\overline{\mathcal{M}}_g})\right),$$

where $\pi : \mathcal{C} \to \overline{\mathcal{M}}_g$ denotes the universal stable curve over Deligne-Mumford's compactification. κ_Δ is used to show that any integral Teichmüller modular form can be extended to a global section on $\overline{\mathcal{M}}_g$.

(2) Let D be the divisor of $\mathcal{M}_g \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ consisting of curves C which have a line bundle L such that $L^{\otimes 2} \cong \Omega_C$ and that $\dim H^0(C,L)$ is positive and even. Then as is shown in [Ty2], $2D$ gives the divisor of $\tau^*(\theta_g)$, and hence a Teichmüller modular form of weight (the weight of θ_g)/2 with divisor D, which exists and is uniquely determined up to constant, is a root of $\tau^*(\theta_g)$ up to constant (see below). Since D is defined over $\overline{\mathbf{Q}}$, a root of $\tau^*(\theta_g)$ times a certain number is defined and primitive over $\mathbf{Z}$. To determine this number, κ_Δ is used as follows: Let A_0, A_Δ, p_{ij} be as in Example 3.1. Then

$$\theta_g(Z) \;=\; 2^{2^{g-1}(2^g-1)} \left(\prod_{\substack{(b_1,\dots,b_g)\,\in\,\{0,1/2\}^g \\ \sum_i b_i \,\in\, \mathbf{Z}}} (-1)^{\sum_i b_i} \right) P \cdot \alpha^2,$$

where

$$\begin{aligned} \alpha \quad &: \quad \text{a primitive element of } \mathbf{Z}\left[q_{ij}^{\pm 1}\ (i \neq j)\right][[q_{11},\dots,q_{gg}]], \\ P \quad &= \quad \prod_{\substack{(b_1,\dots,b_g)\,\in\,\{0,1/2\}^g \\ \sum_i b_i \,\in\, \mathbf{Z}}} \frac{1}{2} \sum_{S \subset \{1,\dots,g\}} (-1)^{\sharp\{k \in S | b_k \neq 0\}} \prod_{i \in S, j \notin S} q_{ij}^{-1/2} \\ &\Rightarrow \quad \left(\text{the constant term of } P|_{q_{ij}=p_{ij}} \in A_\Delta\right)\Big|_{x_1=x_{-2},\dots,x_g=x_{-1}} = 1. \end{aligned}$$

Hence we have (see Exercise 4.5 below):

$$\begin{aligned} &\sqrt{\text{the constant term of } P|_{q_{ij}=p_{ij}}} \in A_0 \\ \Rightarrow \quad &\sqrt{\theta_g|_{q_{ij}=p_{ij}}} \in \begin{cases} \sqrt{-1}\cdot 2^{27} \cdot A_\Delta & (g=3), \\ 2^{2^{g-1}(2^g-1)-1} \cdot A_\Delta & (g \geq 4). \end{cases} \end{aligned}$$

(3) Recall the result of Igusa [Ig2] that the ideal of $S_3^*(\mathbf{C})$ vanishing on the hyperelliptic locus is generated by θ_3. Since the Torelli map $\mathcal{M}_3 \to \mathcal{A}_3$ is dominant and of degree 2, if

we denote ι by the multiplication by -1 on abelian varieties, then

$$\bigoplus_{h\text{: even}} T_{3,h}(\mathbf{C}) = \{ f \in T_3^*(\mathbf{C}) \mid \iota(f) = f \} = S_3^*(\mathbf{C}),$$
$$\bigoplus_{h\text{: odd}} T_{3,h}(\mathbf{C}) = \{ f \in T_3^*(\mathbf{C}) \mid \iota(f) = -f \}.$$

Let f have odd weight. Then by $\iota(f) = -f$, $f = 0$ on the hyperelliptic locus, and hence by Igusa's result, f^2/θ_3 becomes a Siegel modular form. Therefore, $T_3^*(\mathbf{C})$ is generated by $S_3^*(\mathbf{C})$ and $\sqrt{\tau(\theta_3)}$ which implies (3) because $\sqrt{\tau(\theta_3)/N_3}$ is integral and primitive. QED.

Exercise 4.5. Prove that

$$\left(\prod_{\substack{(b_1,\dots,b_g) \in \{0,1/2\}^g \\ \sum_i b_i \in \mathbf{Z}}} (-1)^{\sum_i b_i} \right) = \begin{cases} 1 & (g = 3), \\ -1 & (g \geq 4). \end{cases}$$

TMFs of degree 2. Let k be an algebraically closed field k of characteristic $\neq 2$. Then any proper smooth curve C of genus 2 over k is hyperelliptic, more precisely a base of $H^0(C, \Omega_C)$ gives rise to a morphism $C \to \mathbb{P}_k^1$ of degree 2 ramified at 6 points, and hence

$$\mathcal{M}_2 \otimes_{\mathbf{Z}} k \cong \left\{ (x_1, x_2, x_3 \in \mathbb{P}_k^1 - \{0, 1, \infty\} \mid x_i \neq x_j \ (i \neq j) \right\} / S_6,$$

where each element σ of the symmetric group S_6 degree 6 acts on (x_1, x_2, x_3)'s such as

$$(\sigma(x_1), \sigma(x_2), \sigma(x_3), 0, 1, \infty)$$

is obtained from $\sigma(x_1, x_2, x_3, 0, 1, \infty)$ by some Möbius transformation of $GL_2(k)$. Therefore, $\mathcal{M}_2 \otimes_{\mathbf{Z}} k$ becomes an affine variety, and $T_{2,h}(k) = H^0(\mathcal{M}_2, \lambda^{\otimes h} \otimes_{\mathbf{Z}} k)$ is inifinite dimensional. In fact, it is proved in [I3] that the ring

$$T_2^*(\mathbf{Z}) \stackrel{\text{def}}{=} \bigoplus_{h \in \mathbf{Z}} T_{2,h}(\mathbf{Z})$$

of integral Teichmüller modular forms is generated by $\tau^*(S_2^*(\mathbf{Z}))$ and by $2^{12}/\left(\tau^*(\theta_2)\right)^2$ which is of weight -10.

Construction of TMFs. Assume that $g \geq 3$. Then by results of Mumford [Mu1] and Harer [H1], the **Picard group** of $\mathcal{M}_g$:

$$\mathrm{Pic}(\mathcal{M}_g) \stackrel{\text{def}}{=} \text{the group of linear equivalence classes of line bundles on } \mathcal{M}_g.$$

is isomorphic to $H^2(\mathcal{M}_g(\mathbf{C}),\mathbf{Z}) \cong H^2(\Pi_g,\mathbf{Z})$ (Π_g denotes the Teichmüller modular group of degree g), and this is free of rank 1 generated by the Hodge line bundle λ *over* $\mathbf{Q}$ ($\leftarrow$ can be omitted?). Therefore,

$$\begin{aligned} & D \neq 0 \text{ is an effective divisor on } \mathcal{M}_g \text{ over a subfield } K \text{ of } \mathbf{C} \\ \Rightarrow\ & \text{there are } n, h \in \mathbf{N} \text{ such that } \mathcal{O}_{\mathcal{M}_g}(D)^{\otimes n} \cong \lambda^{\otimes h} \\ \Rightarrow\ & \text{there is } f \in T_{g,h}(K) \text{ such that } (f) = n \cdot D \\ & \text{(for the application, see the proof of Theorem 4.3 (2)),} \end{aligned}$$

$$\begin{aligned} & \mathcal{L} \text{ is a line bundle on } \mathcal{M}_g \otimes_{\mathbf{Z}} K \\ \Rightarrow\ & \text{there are } n, h \in \mathbf{Z} \text{ such that } \mathcal{L}^{\otimes n} \cong \lambda^{\otimes h} \\ \Rightarrow\ & \text{there is } g \in H^0(\mathcal{M}_g \otimes_{\mathbf{Z}} K, \lambda^{\otimes h} \otimes \mathcal{L}^{\otimes -n}) \text{ giving } \mathcal{O}_{\mathcal{M}_g} \xrightarrow{\sim} \lambda^{\otimes h} \otimes \mathcal{L}^{\otimes -n}, \end{aligned}$$

and f, g are uniquely determined by the existence of the Satake-type compactification of $\mathcal{M}_g$. From this method, one can construct Teichmüller modular forms and study their rationality using κ_Δ.

Remark. Morita [Mo] and Mumford [Mu5] conjectured that the stable cohomology groups defined for the moduli spaces of curves over $\mathbf{C}$:

$$\begin{aligned} H^k(\mathcal{M}) &\overset{\text{def}}{=} H^k(\mathcal{M}_g(\mathbf{C}),\mathbf{Q}) = H^k(\Pi_g,\mathbf{Q}) \quad (g \geq 3k-1) \\ &: \quad \text{independent of } g \geq 3k-1 \text{ by Harer's result [H2]} \end{aligned}$$

satisfies that

$$\begin{aligned} \bigoplus_{k \geq 0} H^k(\mathcal{M}) &= \mathbf{Q}[\kappa_1, \kappa_2, ...] : \text{ freely generated over } \mathbf{Q} \\ & \qquad \text{by the } \textbf{tautological classes } \kappa_i = \pi_*\left(\left(c_1\left(\Omega_{\mathcal{C}/\mathcal{M}_g}\right)\right)^{i+1}\right) \end{aligned}$$

(the free generatedness is proved by Miller [M] and Morita [Mo]).

Mumford's isomorphism. Grothendieck-Riemann-Roch's theorem (GRR) says that if $\pi : X \to B$ is a proper morphism over a smooth base, and E is a coherent sheaf on X, then

$$\mathrm{ch}\left(\pi_!(E)\right) \cdot \mathrm{td}(B) \ = \ \pi_*\left(\mathrm{ch}(E) \cdot \mathrm{td}(X)\right)$$

in the Chow ring $\mathrm{CH}^*(B) \otimes_{\mathbf{Z}} \mathbf{Q}$ with $\mathbf{Q}$-coefficients. In order to apply this theorem to a stable curve $\pi : C \to B$ of genus g such that the total space C is smooth, and $E = \Omega_{C/B} \otimes \omega_{C/B}$, put $\gamma = c_1(\Omega_{C/B}) = c_1(\omega_{C/B})$, and let η be the class of the locus in C which consists of ordinary double points of the fibers. Then

$$\frac{\mathrm{td}(C)}{\pi^*\left(\mathrm{td}(B)\right)} \ = \ 1 - \frac{\gamma}{2} + \frac{\gamma^2 + \eta}{12} + \cdots,$$

and hence by GRR,

$$\begin{aligned}
& \mathrm{ch}\left(\pi_*(E)\right) \\
= \;& \pi_*\left(\left(1+\gamma+\frac{\gamma^2}{2}+\cdots\right)^2\cdot(1-\eta+\cdots)\cdot\left(1-\frac{\gamma}{2}+\frac{\gamma^2+\eta}{12}+\cdots\right)\right) \\
= \;& \pi_*\left(1+\frac{3}{2}\gamma+\left(\frac{13}{12}\gamma^2-\frac{11}{12}\eta\right)\right) \\
= \;& (3g-3)+\left(\frac{13}{12}\pi_*(\gamma^2)-\frac{11}{12}\pi_*(\eta)\right) \\
= \;& (3g-3)+(13\lambda-2\pi_*(\eta))
\end{aligned}$$

because by GRR again,

$$\lambda = c_1\left(\pi_*\left(\omega_{C/B}\right)\right) = \pi_*\left(\frac{\gamma^2+\eta}{12}\right).$$

By deformation theory, the cotangent bundle $\mathcal{T}^{\vee}_{\overline{\mathcal{M}}_g}$ of $\overline{\mathcal{M}}_g$ is isomorphic to

$$\begin{aligned}
\pi_*\left(\mathcal{E}xt^1\left(\Omega_{\mathcal{C}/\overline{\mathcal{M}}_g},\mathcal{O}_{\mathcal{C}}\right)^{\vee}\right) &\cong \pi_*\left(\mathcal{E}xt^1\left(\Omega_{\mathcal{C}/\overline{\mathcal{M}}_g}\otimes\omega_{\mathcal{C}/\overline{\mathcal{M}}_g},\omega_{\mathcal{C}/\overline{\mathcal{M}}_g}\right)^{\vee}\right) \\
&\cong \pi_*\left(\Omega_{\mathcal{C}/\overline{\mathcal{M}}_g}\otimes\omega_{\mathcal{C}/\overline{\mathcal{M}}_g}\right) \quad \text{(by Serre's duality)},
\end{aligned}$$

where $\pi:\mathcal{C}\to\overline{\mathcal{M}}_g$ denotes the universal curve over Deligne-Mumford's compactification. Therefore, we have **Mumford's isomorphism** [Mu4]:

$$\bigwedge^{3g-3}\pi_*\left(\mathcal{T}^{\vee}_{\overline{\mathcal{M}}_g}\right) \cong \bigwedge^{3g-3}\pi_*\left(\Omega_{\mathcal{C}/\overline{\mathcal{M}}_g}\otimes\omega_{\mathcal{C}/\overline{\mathcal{M}}_g}\right) \cong \lambda^{\otimes 13}\otimes\mathcal{O}_{\overline{\mathcal{M}}_g}(\overline{\mathcal{M}}_g-\mathcal{M}_g)^{\otimes(-2)}$$

whose section appears as the string amplitude in **String Theory**.

In order to express $\bigwedge^{3g-3}\pi_*\left(\Omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}_g}\right)$ by the Hodge line bundle λ, we consider the morphism

$$\rho_g : S^2\left(\pi_*\left(\Omega_{\mathcal{C}/\mathcal{M}_g}\right)\right) \ni (s,s') \mapsto s\cdot s' \in \pi_*\left(\Omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}_g}\right)$$

between vector bundles on $\mathcal{M}_g$. If $g=2$, then ρ_2 is an isomorphism and gives

$$\lambda^{\otimes 3} \stackrel{\det(\rho_2)}{\longrightarrow} \bigwedge^{3}\pi_*\left(\Omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}_2}\right) \cong \lambda^{\otimes 13} \;\Rightarrow\; \mathcal{O}_{\mathcal{M}_2}\ni 1\mapsto \pm\left(\tau^*(\theta_2)/2^6\right)^2\in\lambda^{\otimes 10},$$

and if $g=3$, then ρ_3 is an isomorphism generically and vanishes on the hyperelliptic locus, hence this gives

$$\lambda^{\otimes 4} \stackrel{\det(\rho_3)}{\longrightarrow} \bigwedge^{6}\pi_*\left(\Omega^{\otimes 2}_{\mathcal{C}/\mathcal{M}_3}\right) \cong \lambda^{\otimes 13} \;\Rightarrow\; \mathcal{O}_{\mathcal{M}_3}\ni 1\mapsto \pm\sqrt{\tau^*(\theta_3)/N_3}\in\lambda^{\otimes 9}.$$

Problem. For $g > 1$, describe a *lift* map:

$$\{\text{SMFs of degree } g-1\} \longrightarrow \{\text{TMFs of degree } g \text{ with level 2 structure}\}$$

obtained as the pullback of the Prym map:

$$\begin{array}{ccc} \{\text{curves of genus } g \text{ with unramified double cover}\} & \longrightarrow & A_{g-1} \\ C' \to C & \longmapsto & \mathrm{Jac}(C')/\mathrm{Jac}(C). \end{array}$$

Problem. Are there Hecke-type operators acting on the space of Teichmüller modular forms? Katsurada pointed that Schottky' J defining the Jacobian locus in $\mathcal{A}_4$ is a Hecke eigenform and is obtained by Ikeda's lift [Ik] from $\Delta(\tau)$ given in Exercise 4.2.

§5. Fundamental groupoid of the moduli space

5.1. Galois group and fundamental groups

The absolute Galois group and cohomology.

The **absolute Galois group** $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$ is the automorphism group of the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ in $\mathbf{C}$. Since $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$ becomes a profinite group given by

$$\varprojlim \mathrm{Gal}(K/\mathbf{Q}) \quad (K \text{ runs through finite extensions of } \mathbf{Q} \text{ in } \overline{\mathbf{Q}}),$$

this is a topological group with Krull topology.

By Grothendieck's theory on l-adic cohomology groups,

$$\begin{aligned} & \text{X is a smooth algebraic variety over } \mathbf{Q} \\ \Rightarrow\ & \mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right) \text{ acts naturally on } \mathbf{Q}_l\text{-coefficient cohomology groups } H^*(X(\mathbf{C}), \mathbf{Q}_l) \\ & \text{i.e., there is a group homomorphism } \mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right) \to \mathrm{Aut}\left(H^*(X(\mathbf{C}), \mathbf{Q}_l)\right) \end{aligned}$$

Assume the existence of the *motivic Galois group* G which is a proalgebraic group over $\mathbf{Q}$ representing (i.e., the fundamental group of) the tannakian category of *motives* over $\mathbf{Q}$, there is a group homomorphism $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right) \to G(\mathbf{Q}_l)$ with Zariski dense image, and hence a certain quotient of $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$ is realized geometrically.

Fundamental groups.

For a smooth algebraic variety X over $\mathbf{Q}$ and points a, b on the associated complex manifold $X(\mathbf{C})$,

$$\begin{aligned} \pi_1(X(\mathbf{C}); a, b) &\stackrel{\text{def}}{=} \{\text{homotopy classes of oriented paths from } a \text{ to } b \text{ on } X(\mathbf{C})\} \\ \pi_1(X(\mathbf{C}); a) &\stackrel{\text{def}}{=} \pi_1(X(\mathbf{C}); a, a) \\ &: \ \text{the fundamental group of } X(\mathbf{C}) \text{ with base point } a. \end{aligned}$$

Then

$$\begin{array}{ccc} \pi_1(X(\mathbf{C}); b, c) \times \pi_1(X(\mathbf{C}); a, b) & \longrightarrow & \pi_1(X(\mathbf{C}); a, c) \\ (\phi, \psi) & \longmapsto & \phi \cdot \psi \stackrel{\text{def}}{=} \overleftarrow{\phi \circ \psi}, \end{array}$$

and hence $\pi_1(X(\mathbf{C}); a, b)$ is a torsor (principally homogeneous space) over $\pi_1(X(\mathbf{C}); a)$ and $\pi_1(X(\mathbf{C}); b)$ under the right and left action respectively.

Let

$$\begin{aligned} \widehat{\pi}_1(X(\mathbf{C}); a) &\stackrel{\text{def}}{=} \varprojlim \pi_1(X(\mathbf{C}); a)/N : \text{the profinite completion of } \pi_1(X(\mathbf{C}); a) \\ & \quad (N \text{ runs through normal subgroups of } \pi_1 \text{ with finite index}), \\ \widehat{\pi}_1(X(\mathbf{C}); a, b) &\stackrel{\text{def}}{=} \text{the profinite completion of } \pi_1(X(\mathbf{C}); a, b) \\ & \quad \text{as a right torsor of } \widehat{\pi}_1(X(\mathbf{C}); a). \end{aligned}$$

Then by Grothendieck's theory on algebraic (etale) fundamental groups,

$$\begin{aligned}\widehat{\pi}_1(X(\mathbf{C});a) &\cong \varprojlim \mathrm{Gal}\left(Y/X_{\overline{\mathbf{Q}}}\right)\\ &\quad (F: Y \to X_{\overline{\mathbf{Q}}} = X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} \text{ runs through finite etale coverings}),\end{aligned}$$

and $\widehat{\pi}_1(X(\mathbf{C});a,b)$ consists of *etale paths* from a to b, i.e., compatible systems of bijections $\gamma_F : F^{-1}(a) \xrightarrow{\sim} F^{-1}(b)$ for finite etale coverings $F : Y \to X_{\overline{\mathbf{Q}}}$ (any element of $\pi_1(X(\mathbf{C});a,b)$ naturally defines an etale path by tracing the fibers of F in $Y(\mathbf{C})$ along the associated paths). Therefore, if a, b are $\mathbf{Q}$-rational points on X, then $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$ acts on $\widehat{\pi}_1(X(\mathbf{C});a,b)$ as

$$(\gamma_F)_F \longmapsto (\sigma \circ \gamma_F \circ \sigma^{-1})_F \quad \left(\sigma \in \mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)\right),$$

and hence there is a group homomorphism $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right) \to \mathrm{Aut}\left(\widehat{\pi}_1(X(\mathbf{C});a,b)\right)$. In his mimeographed note [Gr], Grothendieck posed a program to realize $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$ geometrically by taking X as moduli spaces of curves as follows.

5.2. Teichmüller groupoids

Teichmüller modular groups. For $g, n \geq 0$ such that $3g - 3 + n \geq 0$, Knudsen [K] constructed the moduli stack $\mathcal{M}_{g,n}$ with relative dimension $3g - 3 + n$ over $\mathbf{Z}$ which classifies n-pointed proper smooth curves of genus g. Although $\mathcal{M}_{g,n}$ is only a stack but not a scheme in general, Oda [O] proved that finite etale coverings of $\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ correspond bijectively to normal subgroups with finite index of

$$\begin{aligned}\Pi_{g,n} &\overset{\text{def}}{=} \pi_1\left(\mathcal{M}_{g,n}(\mathbf{C})\right) : \text{the fundamental group of the orbifold } \mathcal{M}_g(\mathbf{C})\\ &= \text{the Teichmüller modular group, or the mapping class group,}\end{aligned}$$

and hence for $a, b \in \mathcal{M}_{g,n}(\mathbf{Q})$,

$$\begin{aligned}\widehat{\pi}_1\left(\mathcal{M}_{g,n}(\mathbf{C});a,b\right) &\overset{\text{def}}{=} \text{the profinite completion of } \pi_1\left(\mathcal{M}_{g,n}(\mathbf{C});a,b\right)\\ &\quad\ \text{as a torsor of the profinite completion } \widehat{\Pi}_{g,n} \text{ of } \Pi_{g,n}\end{aligned}$$

has a natural $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-action.

Caution! To compute exactly Galois and monodromy representations associated with $\Pi_{g,n}$, it is necessary to know this structure and give explicitly $\mathbf{Q}$-rational base points on $\mathcal{M}_{g,n}$. However, the presentation of $\Pi_{g,n}$ (given by Hatcher-Thurston, Wajnryb and Luo [L] using Dehn twists) seems not so simple, and $\mathcal{M}_{g,n}$ seems not to have *natural* $\mathbf{Q}$-rational points.

Esquisse d'un programme. Grothendieck [Gr] introduced the notion of **Teichmüller groupoids** which are defined as the fundamental groupoids of $\mathcal{M}_{g,n}$'s with base points at *infinity* corresponding to maximally degenerate pointed curves. He conjectured that in the category of arithmetic geometry, the system of Teichmüller groupoids (called the **Teichmüller tower**) linked together by fundamental operations (such as *plugging holes, erasing marked points, gluing* and their inverses) behaves like a 2-dimensional complex, i.e., has generators associated with (relative) 1-dimensional objects $\mathcal{M}_{0,4}, \mathcal{M}_{1,1}$ with relations associated with 2-dimensional objects $\mathcal{M}_{0,5}, \mathcal{M}_{1,2}$. Under this conjecture, each element of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is realized as an automorphism of the profinite completion of the Teichmüller tower.

Topology of Teichmüller groupoids. The topological structure of the groupoids is studied in [BK1,2, FG, Fu, HLS, NS, N2], and here we review the formulation and results by H. Nakamura. A **pants decomposition** of a fixed n-pointed Riemann surface of genus g is to decompose it to the union of l-holed and m-pointed Riemann spheres with $l + m = 3$, and then pinching holes to points we have a maximally degenerate n-pointed curve. Nakamura introduced the notion of **quilt decompositions** which are a refinement of pants decomposition by considering 3 seams on each pants and correspond to degenerating behaviors (**Figure**):

$$\begin{aligned}
&\text{the } \textbf{topological Teichmüller groupoid} \text{ for } \mathcal{M}_{g,n} \\
\stackrel{\text{def}}{=}\ &\text{the fundamental groupoid of } \mathcal{M}_{g,n}(\mathbf{C}) \text{ with base points at infinity} \\
&\text{corresponding to maximally degenerate pointed curves} \\
=\ &\left\{ \begin{array}{l} \text{changes of quilts (= pants with seams) decompositions} \\ \text{of a fixed } n\text{-pointed Riemann surface of genus } g \end{array} \right\}.
\end{aligned}$$

Then he proved the following:

Theorem 5.1. ([N2], see [NS, Fu] also) *For $g, n \geq 0$ such that $3g - 3 + n \geq 0$, the **extended Hatcher complex** of type (g, n) is defined as the cell complex whose*

- 0-*cells are isotopy classes of quilt decompositions of a fixed n-pointed Riemann surface of genus g;*
- 1-*cells are the following elementary moves of* 3-*types:*
  - **[F]** ***Fusing (or Associative, A-)moves*** *connecting different sewing processes from two* 3-*holed spheres to one* 4-*holed sphere (***Figure***),*
  - **[S]** ***Simple (or S-)moves*** *connecting different sewing processes from one* 3-*holed spheres to one* 1-*holed real surface of genus* 1 *(***Figure***),*
  - **[D]** ***Dehn half-twists*** *which are half rotations along loops (***Figure***);*
- 2-*cells are relations induced from the basic objects $\mathcal{M}_{0,4}$, $\mathcal{M}_{1,1}$, $\mathcal{M}_{0,5}$ and $\mathcal{M}_{1,2}$ (for example, the **pentagon relation** is induced from $\mathcal{M}_{0,5}$ (***Figure***).*

Then this complex is connected and simply connected. Since the Teichmüller modular group acts on the extended Hatcher complex faithfully, one can see that any topological Teichmüller groupoid is represented as conjectured by Grothendieck.

Sketch of Proof. It is shown in [HLS] and [FG] that the **Hatcher complex** whose 0-cells are isotopy classes of pants decompositions with the above $1, 2$-cells is connected and simply connected. Further, forgetting seams on each quilt we obtain a natural map

$$\text{the extended Hatcher complex} \longrightarrow \text{the Hatcher complex},$$

and the fiber of each pants decomposition is connected and simply connected. Therefore, extended Hatcher complex is also connected and simply connected. QED.

Arithmetic of Teichmüller groupoids. We review an arithmetic of the groupoids realizing a **game of Lego-Teichmüller** given in [Gr]. Here we consider a quilt as a 3-holed $\mathbb{P}^1(\mathbf{C})$ around $0, 1, \infty$ with 3 real lines. Then by gluing holes in several quilts to fit seams to each other (like the Lego game!), we have a real deformation of a maximally degenerate pointed curve (**Figure**). Furthermore, by Theorem 3.3, this deformation can be constructed over the ring consisting of polynomials of moduli parameters and of power series of deformation parameters over $\mathbf{Z}$, and that the elementary moves are described by moving these parameters. Therefore, we have:

Theorem 5.2 ([I5]) *There exists an appropriate base set $\mathcal{L} \subset \mathcal{M}_{g,n}(\mathbf{C})$ of the Teichmüller groupoid for $\mathcal{M}_{g,n}$ consisting of fusing moves and simple moves. For the natural $\mathbf{Z}$-structure of $\mathcal{M}_{g,n}$, $\mathcal{L}$ is a real orbifold of dimension $3g-3+n$ in the real locus, and gives $\mathbf{Z}$-rational tangential base points ($\doteqdot$ unit tangent vectors) around the points at infinity corresponding to maximally degenerate n-pointed curves of genus g.*

If $(g, n) = (0, 4)$, then

$$\mathcal{L} = \mathbf{R} - \{0, 1\} \subset \mathcal{M}_{0,4}(\mathbf{C}) = \mathbb{P}^1(\mathbf{C}) - \{0, 1, \infty\}$$

consists of three fusing moves, and if $(g, n) = (1, 1)$, then

$$\mathcal{L} = \text{the Image of } \left(\sqrt{-1} \cdot \mathbf{R}_{>0}\right) \subset \mathcal{M}_{1,1}(\mathbf{C}) = [H_1/SL_2(\mathbf{Z})]$$

consists of one simple move. For general (g, n), $\mathcal{L} \subset \mathcal{M}_{g,n}(\mathbf{C})$ is constructed by gluing $\mathcal{L}$ in $\mathcal{M}_{0,4}(\mathbf{C})$, $\mathcal{M}_{1,1}(\mathbf{C})$ using the arithmetic Schottky uniformization theory.

Sketch of proof. The construction of fusing moves, which is the main part of the proof, is as follows: Let Δ be a stable graph whose only one vertex v_0 has 4 branches b_i ($1 \leq i \leq 4$), and the other vertices have 3 branches. Further, let Δ' (resp. Δ'') be the trivalent stable graph obtained from Δ by replacing v_0 with an edge having two boundary vertices one of which is a boundary of b_1, b_2 (resp. b_1, b_3) and another is a boundary of b_3, b_4 (resp. b_2, b_4). Then $C_{\Delta'}$ and $C_{\Delta''}$ given in Theorem 3.3 are connected by a fusing move (**Figure**). By the result of Mumford [Mu2] that two Schottky groups over a complete local ring are conjugate if and only if the Mumford curves uniformized by these groups are isomorphic, we compare the moduli and deformation parameters in A_Δ with deformation parameters in $A_{\Delta'}$ and $A_{\Delta''}$. and hence the above fusing move can be constructed by moving parametes in A_Δ appropriately. QED.

5.3. Galois and monodromy representations

Galois representations. The action of Gal $\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$ on profinite Teichmüller groupoids can be described by Theorem 5.2 and Ihara-Anderson's method of Puiseux series [AI, Ih2] as follows:

Theorem 5.3. ([I5]). *Using the base set $\mathcal{L}$ in Theorem 5.2, we can describe the Galois action on all generators of the Teichmüller groupoid for $\mathcal{M}_{g,n}$ as follows:*

- *the action on fusing moves = the action on $\mathcal{L} \subset M_{0,4}$;*
- *the action on simple moves = the action on $\mathcal{L} \subset M_{1,1}$;*
- *the action on Dehn half-twists is given by the* ***cyclotomic character***

$$\chi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \widehat{\mathbf{Z}}^\times \stackrel{\mathrm{def}}{=} \varprojlim (\mathbf{Z}/n\mathbf{Z})^\times; \quad \zeta_n^{\chi(\sigma)} = \sigma(\zeta_n) \ \left(\zeta_n \stackrel{\mathrm{def}}{=} e^{2\pi\sqrt{-1}/n}\right).$$

Example 5.1. Let α (resp. β) be the oriented path around 0 (resp 1) on $\mathcal{M}_{0,4}(\mathbf{C}) = \mathbf{C} - \{0, 1\}$ with tangential base point $\overrightarrow{01}$ (**Figure**). Then α, β are generators of the free profinite group $\widehat{\Pi}_{0,4} = \widehat{\pi}_1(\mathcal{M}_{0,4}(\mathbf{C}); \overrightarrow{01})$ of rank 2, and hence for each $\sigma \in$ Gal $\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$, one can define

$$f_\sigma(\alpha, \beta) \stackrel{\mathrm{def}}{=} (\overrightarrow{01})^{-1} \cdot \sigma(\overrightarrow{01}) \ \in \ \widehat{\Pi}_{0,4}$$

which is, in fact, in the topological commutator subgroup of $\widehat{\Pi}_{0,4}$. Then for a fusing move φ and closed paths a, b on a fixed Riemann surface such that φ changes the quilt decomposition for a to that for b (**Figure**), Theorem 5.3 says that

$$\varphi^{-1} \cdot \sigma(\varphi) \ = \ f_\sigma(\delta_a, \delta_b) \ \left(\sigma \in \mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)\right).$$

Sketch of proof. By Theorem 5.2, there exist formal coordinates $u_0, u_1, ..., u_G$ ($G \stackrel{\mathrm{def}}{=} 3g + n - 4$) over $\mathbf{Z}$ such that for sufficiently small $\varepsilon > 0$,

$$\{(u_0, u_1, ..., u_G \mid 0 < u_0 < 1,\ 0 < u_i < \varepsilon\ (i \geq 1)\}$$

represents the fusing move φ. Let M be the maximal Galois extension of $\overline{\mathbf{Q}}(u_0)$ unramified outside $0, 1, \infty$. Then $\widehat{\Pi}_{0,4} = \widehat{\pi}\left(\mathcal{M}_{0,4}(\mathbf{C}); \overrightarrow{01}\right) \cong \mathrm{Gal}\left(M/\overline{\mathbf{Q}}(u_0)\right)$ acts naturally on M, and for any

$$a = \sum a(n_1, ..., n_G) u_1^{n_1/N} \cdots u_G^{n_G/N} \in M\left[\left[u_1^{n_1/N}, ..., u_G^{n_G/N}\right]\right],$$

we have

$$\begin{aligned}\left(\varphi^{-1} \circ \sigma(\varphi)\right)(a) &= \sum \left((\overrightarrow{01})^{-1} \circ \sigma \circ \overrightarrow{01} \circ \sigma^{-1}\right)(a(n_1, ..., n_G))\, u_1^{n_1/N} \cdots u_G^{n_G/N} \\ &= \sum \left((\overrightarrow{01})^{-1} \circ \overrightarrow{01} \circ f_\sigma\right)(a(n_1, ..., n_G))\, u_1^{n_1/N} \cdots u_G^{n_G/N} \\ &= \sum f_\sigma\left(a(n_1, ..., n_G)\right) u_1^{n_1/N} \cdots u_G^{n_G/N},\end{aligned}$$

where $\overrightarrow{01}(*)$ means the analytic continuation of $*$ along $\overrightarrow{01}$. Therefore, $\varphi^{-1} \cdot \sigma(\varphi) = f_\sigma\left(\delta_a, \delta_b\right)$. QED.

Grothendieck-Teichmüller group. Belyi [B] proved that any proper smooth curve C over $\overline{\mathbf{Q}}$ can be realized as a finite covering (Belyi's covering) $f : C \to \mathbb{P}^1$ over $\mathbf{C}$ unramified outside $0, 1, \infty$, hence corresponds to a subgroup Γ_f of $\Pi_{0,4}$ of finite index. Using this fact, he showed that

$$\begin{aligned} & \sigma \in \mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right),\ \sigma \neq \mathrm{Id}_{\overline{\mathbf{Q}}} \\ \Rightarrow\ & \text{there is a } J \in \overline{\mathbf{Q}} \text{ such that } \sigma(J) \neq J \\ \Rightarrow\ & E_J \overset{\text{def}}{=} \text{the elliptic curve } y^2 = 4x^3 - \frac{3J}{J-1}x - \frac{J}{1-J} \text{ with } j\text{-invariant } J \text{ (p.18)} \\ & \text{is not isomorphic to } E_{\sigma(J)} \text{ over } \mathbf{C} \\ \Rightarrow\ & \Gamma_f \text{ is not conjugate to } \Gamma_{f'}\ (f, f' \text{ are Belyi's coverings of } E_J, E_{\sigma(J)} \text{ respectively}) \\ \Rightarrow\ & \text{the outer action of } \sigma \text{ on } \widehat{\Pi}_{0,4} \text{ is not trivial,}\end{aligned}$$

which implies the injectivity of the map:

$$\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right) \ni \sigma \longmapsto (\chi(\sigma), f_\sigma) \in \widehat{\mathbf{Z}}^\times \times \widehat{\Pi}_{0,4}.$$

Drinfeld [Dr] introduced the profinite **Grothendieck-Teichmüller group** $\widehat{GT}$ as a subgroup of $\widehat{\mathbf{Z}}^\times \times \widehat{F}_2'$ ($\widehat{F}_2'$ denotes the topological commutator subgroup of the free profinite group $\widehat{F}_2$ generated by x, y) consisting of (λ, f) which satisfy

- $f(x,y) \cdot f(y,x) = 1$ which follows from the relation $\overrightarrow{10} \cdot \overrightarrow{01} = \mathrm{Id}$;
- $f(z,x) \cdot z^m \cdot f(y,z) \cdot y^m \cdot f(x,y) \cdot x^m = 1$, if $xyz = 1$, $m = (\lambda - 1)/2$ which follows from the relation between $\overrightarrow{01}$, $\overrightarrow{1\infty}$, $\overrightarrow{\infty 0}$ in $\mathcal{M}_{0,4}$;
- $f(x_{12}, x_{23}) \cdot f(x_{34}, x_{45}) \cdot f(x_{51}, x_{12}) \cdot f(x_{23}, x_{34}) \cdot f(x_{45}, x_{51}) = 1$ in $\widehat{\Pi}_{0,5}$ (for the definition of x_{ij}, see [Ih1]) which follows from the pentagon relation on $\mathcal{M}_{0,5}$.

It is shown by Drinfeld and Ihara that $\widehat{GT}$ is regarded as the automorphism group of the profinite Teichmüller tower of genus 0, and hence contains $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$.

Let $\Gamma(2)$ be the congruence subgroup of $SL_2(\mathbf{Z})$ of level 2 (p.6). Then the geometric quotient $H_1/\Gamma(2)$ is the moduli space of elliptic curves over $\mathbf{C}$ with symplectic level 2 structure modulo ± 1, and the λ-function:

$$\begin{array}{rcccl} \lambda & : & H_1/\Gamma(2) & \longrightarrow & \mathcal{M}_{0,4}(\mathbf{C}) = \mathbb{P}^1(\mathbf{C}) - \{0,1,\infty\} \\ & & \tau & \longmapsto & \dfrac{\wp_{\mathbf{Z}+\mathbf{Z}\tau}(\tau/2) - \wp_{\mathbf{Z}+\mathbf{Z}\tau}((\tau+1)/2)}{\wp_{\mathbf{Z}+\mathbf{Z}\tau}(\tau/2) - \wp_{\mathbf{Z}+\mathbf{Z}\tau}(1/2)} = \mathbf{16}\ e^{\pi\sqrt{-1}\tau} + \cdots \end{array}$$

gives Legendre's model of $\pi_1(\mathcal{M}_{0,4})$ (see [N1]) connecting simple and fusing moves:

$$\begin{array}{rcccc} \lambda_* & : & \pi_1\left(H_1/\Gamma(2); \overrightarrow{i\infty\, 0}\right) & \longrightarrow & \pi_1\left(\mathcal{M}_{0,4}(\mathbf{C}); \overrightarrow{01}\right) \\ & & \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} & \longmapsto & \alpha, \\ & & \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} & \longmapsto & \beta. \end{array}$$

Using this model, Lochack, Nakamura and Schneps [LNS, NS] translated the relations for $\mathcal{M}_{1,1}, \mathcal{M}_{1,2}$ to those for $\mathcal{M}_{0,4}, \mathcal{M}_{0,5}$ and they introduced a subgroup $\mathbb{\Gamma}$ satisfying that

- $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right) \subset (\overset{?}{=})\ \mathbb{\Gamma} \subset (\overset{?}{=})\ \widehat{GT} \subset \widehat{\mathbf{Z}}^{\times} \times \widehat{\Pi}_{0,4}$,
- $\mathbb{\Gamma}$ acts on profinite Teichmüller modular groups extending the Galois action for tangential base points of restricted types.

Exercise 5.1. Show that $\lambda(\tau) = 16 e^{\pi\sqrt{-1}\tau} + \cdots$.

This result together with Theorem 5.3 imply the simple picture:

$\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right) \subset \mathbb{\Gamma} =$ Aut (the profinite Teichmüller tower)
giving the Galois action on each Teichmüller groupoid.

Monodromy representations. **Conformal field theory** (CFT) provides vector bundles $(\mathcal{V}, \nabla)$ with projectively flat connection over the moduli of pointed Riemann surfaces with first-order infinitesimal structure on each marked point. This theory was studied by Moore-Seiberg [MS] as a representation theory of Teichmüller groupoids by the Riemann-Hilbert correspondence, and constructed rigorously by Tsuchiya-Ueno-Yamada [TUY]. Then in a similar way to the proof of Theorem 5.3, we have

Theorem 5.4. ([I6]) *Using the base set $\mathcal{L}$, we can compute the monodromy representation of any Teichmüller groupoid for the TUY-theory as:*

- *monodromy of fusing moves*
  $=$ *connection matrices of the Knizhnik-Zamolodchikov differential equation;*

- *monodromy of simple moves*
  $=$ *transformation matrices of non-abelian theta functions given in [KP];*

- *monodromy of Dehn half-twists*
  $= \exp\left(\pi\sqrt{-1} \times (\textit{the residues of the connection forms})\right)$.

Hence the monodromy for TUY is given as the monodromy for the Wess-Zumino-Witten model given by Kohno [Ko].

Example 5.2. Let V be a vector space over $\mathbf{C}$, and $A, B \in \mathrm{End}(V)$ be linear endomorphisms of V. We consider the linear differential equation on $t \in (0,1)$:

$$G'(t) \;=\; \left(\frac{A}{t} + \frac{B}{t-1}\right) G(t),$$

and let $G_i(t)$ $(i = 1, 2)$ be its two solutions satisfying the asymptotic condition:

$$\lim_{t\to 0} \frac{G_0}{t^A} = \lim_{t\to 1} \frac{G_1}{(1-t)^B} = \mathrm{id}_V.$$

Then

$$\Phi(A,B) \overset{\text{def}}{=} G_1(t)^{-1} \cdot G_0(t) : \text{ the } \textbf{connection matrix}$$

is independent of t and hence is an automorphism of V. Further, using iterated integrals $\int \omega \cdots \omega$ of $\omega \overset{\text{def}}{=} \left(\frac{A}{t} + \frac{B}{t-1}\right) dt$, we have

$$\begin{aligned}
\Phi(A,B) \;&=\; \lim_{\varepsilon\to 0} \left\{ \varepsilon^{-B} \left(\sum_{n=0}^{\infty} \int_{1-\varepsilon}^{t} \underbrace{\omega \cdots \omega}_{n} \right)^{-1} \left(\sum_{n=0}^{\infty} \int_{\varepsilon}^{t} \underbrace{\omega \cdots \omega}_{n} \right) \varepsilon^{A} \right\} \\
&=\; \lim_{\varepsilon\to 0} \left\{ \varepsilon^{-B} \left(\sum_{n=0}^{\infty} \int_{\varepsilon}^{1-\varepsilon} \underbrace{\omega \cdots \omega}_{n} \right) \varepsilon^{A} \right\}.
\end{aligned}$$

Let $(\mathcal{V}, \nabla)$ be as above, and let φ be a fusing move and a, b be closed paths on a fixed Riemann surface such that φ changes the quilt decompostion for a to that for b (**Figure**). Then the monodromy of φ with respect to $(\mathcal{V}, \nabla)$ is given by

$$\Phi\left(\mathrm{Res}_a(\nabla), \mathrm{Res}_b(\nabla)\right) \mod(\mathbf{C}^{\times}),$$

where $\mathrm{Res}_*(\nabla)$ is the principal part of the connection form of ∇ around $*$.

5.4. Motivic theory

A joke!? Main targets in number theory and in physics are respectively:

$\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$		**UNIVERSE**
Grothendieck's program $\nwarrow$		$\nearrow$ string theory
	Moduli of curves	

Roughly speaking, these objects correspond to:

Etale aspect	**Motivic theory?**	De Rham aspect
$\nwarrow$		$\nearrow$
	π_1 of the Moduli	

Motif on $\pi_1(\mathcal{M}_{0,4})$. For a smooth algebraic variety X over $\mathbf{Q}$ with $\mathbf{Q}$-rational (tangential) point a, in order to *linearlize*

$$\pi_1 \stackrel{\text{def}}{=} \pi_1\left(X(\mathbf{C});a\right),$$

put

$$\begin{aligned}
\mathbf{Q}\left[\pi_1\right] &\stackrel{\text{def}}{=} \left\{\text{finite sums } \sum_i \alpha_i g_i \;\middle|\; \alpha_i \in \mathbf{Q}, g_i \in \pi_1\right\} : \text{ the group } \mathbf{Q}\text{-algebra of } \pi_1,\\
J &\stackrel{\text{def}}{=} \left\{\sum_i \alpha_i g_i \in \mathbf{Q}\left[\pi_1\right] \;\middle|\; \sum_i \alpha_i = 0\right\} : \text{ the augmentation ideal of } \mathbf{Q}\left[\pi_1\right],\\
\mathbf{Q}\left[\pi_1\right]^{\wedge} &\stackrel{\text{def}}{=} \varprojlim \mathbf{Q}\left[\pi_1\right]/J^n : \text{ the completed group algebra of } \pi_1,\\
\mathcal{G}\left[\pi\right] &\stackrel{\text{def}}{=} \left\{M \in \mathbf{Q}\left[\pi_1\right]^{\wedge} \;\middle|\; \Delta(M) = 1\otimes M + M \otimes 1\right\} \; (\Delta : \text{ the diagonal map})\\
&: \text{ the Malcev Lie algebra associated with } \pi_1 \text{ for } [M,N] \stackrel{\text{def}}{=} MN - NM.
\end{aligned}$$

In particular,

$$\begin{aligned}
&X = \mathcal{M}_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}, \quad a = \overrightarrow{01}\\
&\mathbf{Q}\langle\langle A,B\rangle\rangle \stackrel{\text{def}}{=} \text{the ring of noncommutative power series over } \mathbf{Q} \text{ of } A,B\\
&\Rightarrow \quad \mathbf{Q}\left[\pi_1\left(X(\mathbf{C});\overrightarrow{01}\right)\right]^{\wedge} \cong \mathbf{Q}\langle\langle A,B\rangle\rangle\\
&\qquad\qquad \alpha \quad \leftrightarrow \quad e^A \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{A^n}{n!}\\
&\qquad\qquad \beta \quad \leftrightarrow \quad e^B,
\end{aligned}$$

and

$$\left\{\begin{array}{lll} f_\sigma\left(e^A, e^B\right) & \in & \mathbf{Q}_l\langle\langle A,B\rangle\rangle,\\ \Phi\left(A,B\right) & \in & \mathbf{R}\langle\langle A,B\rangle\rangle \end{array}\right.$$

describe the Galois, Hodge counterparts of the mixed Tate motives on $\pi_1(\mathcal{M}_{0,4})$ respectively (see [D2]).

Galois counterpart. In order to compute f_σ, as a special case of the computation of l-adic polylogarithms in [NW], put

$$c^{1/m} \stackrel{\text{def}}{=} \exp(\log(c)/m) \text{ for } c \in \mathbf{C}^\times \text{ and the principal branch of log,}$$

$$\kappa_{l^n}^{(a)} : \mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right) \to \mathbf{Z}_l \text{ defined by } \sigma\left(\left(1-\zeta_{l^n}^{\chi(\sigma)^{-1}\cdot a}\right)^{1/l^m}\right) = \zeta_{l^m}^{\kappa_{l^n}^{(a)}(\sigma)} \cdot (1-\zeta_{l^n}^{a})^{1/l^m},$$

$$H_{l^n} \stackrel{\text{def}}{=} \text{the kernel of the homomorphism } \widehat{\Pi}_{0,4} \to \mathbf{Z}/l^n\mathbf{Z} \text{ sending } \alpha \mapsto 1,\ \beta \mapsto 0.$$

Then H_{l^n} is a free profinite group generated by $\alpha^{l^n}, \alpha^{-a}\beta\alpha^{a}$ $(a = 0, ..., l^n-1)$, and contains $\widehat{F}_2'$. Since

$$\begin{array}{cl}
 & \left(1-\zeta_{l^n}^{b}\cdot t^{1/l^n}\right)^{1/l^m} \\
\stackrel{\sigma^{-1}}{\longmapsto} & \left(1-\zeta_{l^n}^{\chi(\sigma)^{-1}\cdot b}\cdot t^{1/l^n}\right)^{1/l^m} \\
\stackrel{\overrightarrow{01}}{\longmapsto} & \left(1-\zeta_{l^n}^{\chi(\sigma)^{-1}\cdot b} - \zeta_{l^n}^{\chi(\sigma)^{-1}\cdot b}\cdot \sum_{k=1}^{\infty}\begin{pmatrix} 1/l^n \\ k \end{pmatrix}(t-1)^k\right)^{1/l^m} \\
\stackrel{\sigma}{\longmapsto} & \zeta_{l^m}^{\kappa_{l^n}^{(b)}(\sigma)}\left(1-\zeta_{l^n}^{b} - \zeta_{l^n}^{b}\cdot \sum_{k=1}^{\infty}\begin{pmatrix} 1/l^n \\ k \end{pmatrix}(t-1)^k\right)^{1/l^m} \\
\stackrel{(\overrightarrow{01})^{-1}}{\longmapsto} & \zeta_{l^m}^{\kappa_{l^n}^{(b)}(\sigma)}\left(1-\zeta_{l^n}^{b}\cdot t^{1/l^n}\right)^{1/l^m},
\end{array}$$

and

$$\left(\alpha^{-a}\circ\beta\circ\alpha^{a}\right)\left(1-\zeta_{l^n}^{b}\cdot t^{1/l^n}\right)^{1/l^m} = \begin{cases} \left(1-\zeta_{l^n}^{b}\cdot t^{1/l^n}\right)^{1/l^m} & (a\neq b), \\ \zeta_{l^m}^{-1}\left(1-\zeta_{l^n}^{b}\cdot t^{1/l^n}\right)^{1/l^m} & (a=b), \end{cases}$$

we have

$$f_\sigma(\alpha,\beta) \equiv \prod_{0\le a<l^n}\left(\alpha^{-a}\beta\alpha^{a}\right)^{-\kappa_{l^n}^{(a)}} \mod [H_{l^n}, H_{l^n}].$$

Therefore, using $\log\left(e^{A}e^{B}e^{-A}\right) = \sum_{i=0}^{\infty}\frac{(\mathrm{ad}A)^i(B)}{i!}$,

$$\chi_m : \mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right) \to \mathbf{Z}_l \text{ defined by } \chi_m(\sigma) \equiv \sum_{a=0}^{l^n-1} a^{m-1}\cdot\kappa_{l^n}^{(a)}(\sigma) \bmod (l^n)$$

$$\Rightarrow\quad \log\left(f_\sigma\left(e^{A},e^{B}\right)\right) \equiv \sum_{i=0}^{\infty}\frac{(-1)^i\chi_{i+2}(\sigma)}{(i+1)!}[\underbrace{A,...,[A,[A}_{i+1},B]],...],$$

and for each odd integer $m \geq 3$, Soulé proved that $\chi_m(\sigma)$ gives a nonzero element of $\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})}(\mathbf{Q}_l, \mathbf{Q}_l(m))$ ($(1 - l^{m-1})\chi_m(\sigma)$ is called m-th **Soulé's character** usually). Hain-Matsumoto [HM] (see also [DG]) proved Deligne-Ihara's conjecture [D2, Ih1] that the Malcev Lie algebra associated with the Galois structure on $\pi_1(\mathcal{M}_{0,4})$ are generated by Soulé's characters.

Hodge counterpart. To compute $\Psi(A, B)$ as an element of $\mathbf{R}\langle\langle A, B\rangle\rangle$, following [LM], let x, y be variables commuting each other and with A, B, and define an $\mathbf{R}$-linear map

$$\begin{array}{rclcl} \Psi : \mathbf{R}\langle\langle A, B\rangle\rangle & \to & \mathbf{R}\langle\langle A, B\rangle\rangle[[x, y]] & \to & \mathbf{R}\langle\langle A, B\rangle\rangle \\ H(A, B) & \mapsto & H(A - x, B - y) & & \\ & & y^p H x^q & \mapsto & B^p H A^q \end{array}$$

such that $\Psi\left(\sum_{n=0}^{\infty} H_n\right) = \sum_{n=0}^{\infty} \Psi(H_n)$ for $H_n \in \mathbf{R}\langle\langle A, B\rangle\rangle$ of total degree n. Then

$$\Psi(BH) = \Psi(HA) = 0 \ (H \in \mathbf{R}\langle\langle A, B\rangle\rangle), \quad \Psi(\Phi(A, B)) = \Phi(A, B)$$

because

$$\Phi(A - x, B - y) = \left(t^{-x}(1 - t)^{-y} G_1(t)\right)^{-1} \cdot \left(t^{-x}(1 - t)^{-y} G_0(t)\right) = \Phi(A, B).$$

Therefore, we have

$$\begin{aligned} \Phi(A, B) &= \Psi(\Phi(A, B)) \\ &= \lim_{\varepsilon \to 0} \Psi\left(\varepsilon^{-B}\left(\sum_{n=0}^{\infty} \int_{\varepsilon}^{1-\varepsilon} \underbrace{\omega \cdots \omega}_{n}\right) \varepsilon^{A}\right) \quad \left(\omega \overset{\text{def}}{=} \left(\frac{A}{t} + \frac{B}{t - 1}\right) dt\right) \\ &= \lim_{\varepsilon \to 0} \Psi\left(\sum_{n=0}^{\infty} \int_{\varepsilon}^{1-\varepsilon} \underbrace{\omega \cdots \omega}_{n}\right) \\ &\overset{(*1)}{=} 1 - \zeta(2)[A, B] - \zeta(3)[A, [A, B]] - \zeta(3)[B, [A, B]] \\ &\quad - \zeta(4)[A, [A, [A, B]]] - \zeta(4)[B, [B, [A, B]]] \\ &\quad - \zeta(1, 3)[A, [B, [A, B]]] - \frac{\zeta(2)^2}{2}[A, B]^2 + \cdots \\ &\quad \text{(by Drinfeld? Kontsevich?)}, \end{aligned}$$

which is called **Drinfeld's associator**, where for integers $k_1, ..., k_m$ such that $k_m \geq 2$,

$$\begin{aligned} \zeta(k_1, ..., k_m) &\overset{\text{def}}{=} \sum_{1 \leq a_1 < \cdots < a_m} \frac{1}{a_1^{k_1} \cdots a_m^{k_m}} : \ \text{the } \textbf{multiple zeta values} \\ &\overset{(*2)}{=} \underbrace{\int_0^1 \frac{dt}{t} \cdots \int_0^t \frac{dt}{t}}_{k_m - 1} \int_0^t \frac{dt}{1 - t} \cdots \underbrace{\int_0^t \frac{dt}{t} \cdots \int_0^t \frac{dt}{t}}_{k_1 - 1} \int_0^t \frac{dt}{1 - t}. \end{aligned}$$

Exercise 5.2. Prove the above $(*1)$ and $(*2)$.

Goncharov [Go] and Terasoma [Te] (see also [DG]) proved Deligne-Zagier's conjecture [D2] that the Malcev Lie algebra associated with the Hodge structure on $\pi_1(\mathcal{M}_{0,4})$ are generated by $\zeta(m)$ ($m \geq 3$, odd) which implies the following:

$$
\begin{aligned}
& L_w \overset{\text{def}}{=} \left\{ \sum \mathbf{Q} \cdot \zeta(k_1, ..., k_m) \in \mathbf{R} \;\middle|\; k_1 + \cdots + k_m = w \right\}, \\
& d_w \text{ are the integers defined by } \sum_{n=0}^{\infty} d_n t^n = \frac{1}{1 - t^2 - t^3} \\
\Rightarrow \quad & \dim_{\mathbf{Q}} L_w \leq d_w.
\end{aligned}
$$

Caution! If $X = \mathcal{M}_{1,1}$ which is not an algebraic variety, then

$$
\begin{aligned}
& \pi_1(X(\mathbf{C})) \cong SL_2(\mathbf{Z}) \text{ is generated by the 2 elements} \\
& \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \text{ of finite orders} \\
\Rightarrow \quad & \text{there is } m \in \mathbf{N} \text{ such that } \sigma_i^m = 1 \ (i = 1, 2) \\
\Rightarrow \quad & m \log(\sigma_i) = \log(\sigma_i^m) = -\sum_{n=1}^{\infty} \frac{(1 - \sigma_i^m)^n}{n} = 0 \text{ in } \mathbf{Q}\left[SL_2(\mathbf{Z})\right]^{\wedge} \\
\Rightarrow \quad & \log(\sigma_i) = 0, \text{ i.e., } \sigma_i = 1 \text{ in } \mathbf{Q}\left[SL_2(\mathbf{Z})\right]^{\wedge} \ (i = 1, 2) \\
\Rightarrow \quad & \mathbf{Q}\left[SL_2(\mathbf{Z})\right]^{\wedge} = \{0\}.
\end{aligned}
$$

However, using Legendre's model and the results of [HM, Go, Te, DG], one can construct a mixed Tate motif on $\mathbf{Q}\left[\Gamma(2)\right]^{\wedge}$ which is generated by $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Q}(m))$ ($m \geq 3$, odd), and by $\mathrm{Ext}^1(\mathbf{Q}, \mathbf{Q}(1))$ corresponding to $\left\{ \sqrt[n]{16} \right\}_{n \in \mathbf{N}}$ because the first Fourier coefficient of $\lambda(\tau)$ is 16.

Exercise 5.3. Prove that

- $SL_2(\mathbf{Z})$ is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ (cf. Exercise 3.1);
- $\Gamma(2)$ is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Problem. Using Theorem 5.3, construct a motivic theory on $\mathbf{Q}\left[\Pi'_{g,n}\right]$, where $\Pi'_{g,n}$ denotes the principal subgroup of $\pi_1(\mathcal{M}_{g,n}(\mathbf{C})) = \Pi_{g,n}$ of level 2. Does this provide mixed Tate motives generated by $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Q}(m))$ ($m \geq 3$, odd), and by $\mathrm{Ext}^1(\mathbf{Q}, \mathbf{Q}(1))$ corresponding to $\left\{ \sqrt[n]{2} \right\}_{n \in \mathbf{N}}$?

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